Phan 1 (Toan cao cap: C1): (3 ñiểm)
Chöông 1: Giải tích: Ham nhieu bien.
Chöông 2: Tích phan boi hai.

Phan 2 (Toan cao cap: C2): (3 ñiểm)
Chöông 1: Ma tran.
Chöông 2: Ñònh thöc.
Chöông 3: KGVT.

Phan 3 (XS+TK): (4 ñiểm)
Chöông 1: Phan phoi nhò thöc, phan phoi chuan.
Chöông 2: Vecto ngau nhien.
Chöông 3: Öôc löông.

Phaàn 1:

Chöông 1

BAØI 1: ÑAÏO HAØM - VI PHAÂN -ÖÙNG DUÏNG

0. Ñaïo haøm rieâng:

0.1 Ñaïo haøm rieâng baäc cao:

0.1 Ñaïo haøm hoãn hôïp

Ñònh lyù (Schwarz):

Giaû söû z'_x, z'_y z''_{xy} toàn taïi vaø lieân tuïc thì coù z''_{yx} vaø $z''_{xy} = z''_{yx}$

1. Cöïc trò töï do haøm 2, 3 biến:

Ñònh lyù veà ñieàu kieän caàn: f = f(x,y) hay f = f(x, y, z) coù cöïc trò ñòa phöông taïi $M_0(x_0, y_0)$ $(M_0(x_0, y_0, z_0))$ vaø caùc ñaïo haøm rieâng toàn taïi ==> $M_0(x_0,y_0)$ $(M_0(x_0, y_0, z_0))$ laø ñieåm döøng.

Ñònh lyù veà ñieàu kieän ñuû: Neáu $M_0(x_0,y_0)$ laø ñieåm döøng, thì chöùa chaéc $M_0(x_0,y_0)$ laø ñieåm cöïc trò, maø caàn phaûi coù theâm nhöõng ñieàu kieän cuï theå veà ñaïo haøm baäc hai nhö phöông phaùp sau:

Hai biến:

1/ $H_1 = f''_{xx} > 0,\ H_2 = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{vmatrix} > 0$ ==> f ñaït cöïc tieåu taïi X_0.

2/ $H_1 = f''_{xx} < 0,\ H_2 = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{vmatrix} > 0$ ==> f ñaït cöïc ñaïi taïi X_0.

Ba biến:

• Cöïc tieåu: (döông caû) $H_1 = f''_{xx} > 0,\ H_2 = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{vmatrix} > 0,\ H_3 = \begin{vmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{vmatrix} > 0$

• Cöïc ñaïi: (ñan daáu, baét ñaàu töø aâm)

$$H_1 = f''_{xx} < 0,\ H_2 = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{vmatrix} > 0,\ H_3 = \begin{vmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{vmatrix} < 0$$

Ñieàu kieän ñuû daïng vi phaân nhieàu bieán toång quaùt: $M_0(x_0, y_0, z_0))$ laø ñieåm döøng, xeùt daïng vi phaân $d^2 f(X_0)$:

a/ Laø xaùc ñònh döông $\left(d^2 f(X_0) > 0\right)$ thì f ñaït cöïc tieåu taïi X_0.

b/ Laø xaùc ñònh aâm $\left(d^2 f(X_0) < 0\right)$ thì f ñaït cöïc ñaïi taïi X_0.

c/ $\left(d^2 f(X_0) = 0\right)$ thì taïi X_0 chöùa coù keát luaän. Phaûi xeùt vi phaân caáp 3...

Vi phaân caáp 2: f=f(x,y)

$$d^2 f = f''_{x^2} dx^2 + 2 f''_{xy} dxdy + f''_{y^2} dy^2$$

Vi phan cap 2: f=f(x,y,z)

$$d^2f = f''_{x^2}dx^2 + f''_{y^2}dy^2 + f''_{z^2}dz^2 + 2f''_{xy}dxdy + 2f''_{xz}dxdz + 2f''_{yz}dydz$$

2. Cöc trò vöông ham 2 biến co 1 rang buoc: Cöc trò $f = f(x_1, x_2)$ va 1 rang buoc $j(x_1, x_2) = 0$

Ham Lagrange: $f(x_1, x_2, l) = f(x_1, x_2) + l\, j(x_1, x_2)$

Ñieu kien can: $f = f(x_1, x_2)$ ñat cöc trò tai M_0 thì M_0 thoa $\overset{uuu}{Ñ}f = 0$ (giai he phöông trình nay ñe tìm M_0 la ñiem döng cua ham $f(x_1, x_2, l)$).

Ñònh ly ve ñieu kien ñu:

i/ $H_2 < 0$ Þ thì f ñat cöc tieu.

ii/ $H_2 > 0$ Þ thì f ñat cöc ñai.

Trong ño: $H_2 = \left|\begin{array}{c|cc} 0 & j'_x & j'_y \\ \hline j'_x & f''_{x_1x_1} & f''_{x_1x_2} \\ j'_y & f''_{x_2x_1} & f''_{x_2x_2} \end{array}\right| = \left|\begin{array}{c|cc} f''_{ll} & f''_{lx_1} & f''_{lx_2} \\ \hline f''_{lx_1} & f''_{x_1x_1} & f''_{x_1x_2} \\ f''_{lx_2} & f''_{x_2x_1} & f''_{x_2x_2} \end{array}\right|$

Ñieu kien ñu dang vi phan tong quat cho ham Lagrange $\underline{f}$:

$f(x_1, x_2, l) = f(x_1, x_2) + l\, g(x_1, x_2)$, xet dang vi phan $d^2f(X_0)$:

a/ La xac ñònh döông $\left(d^2f(X_0) > 0\right)$ thì f ñat cöc tieu tai X_0.

b/ La xac ñònh am $\left(d^2f(X_0) < 0\right)$ thì f ñat cöc ñai tai X_0.

c/ $\left(d^2f(X_0) = 0\right)$ thì tai X_0 chöa co ket luan. Phai xet vi phan cap 3...

- Ta tính ham 2 bien: L(x, y, λ) = f(x, y) + λ φ (x, y)

$d^2L = L''_{x^2}dx^2 + 2L''_{xy}dxdy + L''_{y^2}dy^2$. (*)

- Chu y: $j'_x dx + j'_y dy = 0$, $dx^2 + dy^2 > 0$
- Neu: $d^2L(x_0,y_0) > 0$ thì (x_0,y_0) la ñiem cöc tieu.
  $d^2L(x_0,y_0) < 0$ thì (x_0,y_0) la ñiem cöc ñai.
  $d^2L(x_0,y_0) = 0$ thì (x_0,y_0) khong la ñiem cöc trò.
- Töông tö ham 3 bien: L(x,y,z, λ) = f(x,y,z) + λ φ (x,y,z)
  Thì $d^2L = L''_{x^2}dx^2 + L''_{y^2}dy^2 + L''_{z^2}dz^2 + 2(L''_{xy}dxdy + L''_{xz}dxdz + L''_{yz}dydz)$

3. Cöc trò vöông ham 3 biến co 1 rang buoc:

Cöc trò $f = f(x_1, x_2, x_3)$ va rang buoc $g(x_1, x_2, x_3) = 0$

Ham Lagrange: $f(x_1, x_2, x_3, l) = f(x_1, x_2, x_3) + l\, g(x_1, x_2, x_3)$

Ñieu kien ñu dang vi phan tong quat ham Lagrange $\underline{f}$:

$f(x_1, x_2, x_3, l) = f(x_1, x_2, x_3) + l\, g(x_1, x_2, x_3)$, xet dang vi phan $d^2f(X_0)$:

a/ La xac ñònh döông $\left(d^2f(X_0) > 0\right)$ thì f ñat cöc tieu tai X_0.

b/ La xac ñònh am $\left(d^2f(X_0) < 0\right)$ thì f ñat cöc ñai tai X_0.

c/ $\left(d^2 f\left(X_0\right)=0\right)$ thì tai X_0 chöa co ket luan. Phai xet vi phan cap 3...

<u>Ñieu kien ñu ve ñònh thöc Hess, xet 2 ñònh thöc:</u>

i/ $H_2 < 0$, $H_3 < 0 \Rightarrow$ thì f ñat cöc tieu.

ii/ $H_2 > 0$, $H_3 < 0 \Rightarrow$ thì f ñat cöc ñai.

$$H_2=\left|\begin{array}{c|cc} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \hline \frac{\partial g}{\partial x_1} & f''_{x_1x_1} & f''_{x_1x_2} \\ \frac{\partial g}{\partial x_2} & f''_{x_2x_1} & f''_{x_2x_2} \end{array}\right| = \left|\begin{array}{c|cc} f''_{ll} & f''_{lx_1} & f''_{lx_2} \\ \hline f''_{lx_1} & f''_{x_1x_1} & f''_{x_1x_2} \\ f''_{lx_2} & f''_{x_2x_1} & f''_{x_2x_2} \end{array}\right|,$$

$$H_3=\left|\begin{array}{c|ccc} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \\ \hline \frac{\partial g}{\partial x_1} & f''_{x_1x_1} & f''_{x_1x_2} & f''_{x_1x_3} \\ \frac{\partial g}{\partial x_2} & f''_{x_2x_1} & f''_{x_2x_2} & f''_{x_2x_3} \\ \frac{\partial g}{\partial x_3} & f''_{x_3x_1} & f''_{x_3x_2} & f''_{x_3x_3} \end{array}\right| = \left|\begin{array}{c|ccc} f''_{ll} & f''_{lx_1} & f''_{lx_2} & f''_{lx_3} \\ \hline f''_{lx_1} & f''_{x_1x_1} & f''_{x_1x_2} & f''_{x_1x_3} \\ f''_{lx_2} & f''_{x_2x_1} & f''_{x_2x_2} & f''_{x_2x_3} \\ f''_{lx_3} & f''_{x_3x_1} & f''_{x_3x_2} & f''_{x_3x_3} \end{array}\right|$$

4. Cöc trò vöông ham 3 biến co 2 rang buoc: Cöc trò $f = f(x_1, x_2, x_3)$ va 2 rang buoc

$$\begin{cases} g_1(x_1,x_2,x_3)=0 \\ g_2(x_1,x_2,x_3)=0 \end{cases}.$$

$$f(x_1,x_2,x_3,l_1,l_2)= f(x_1,x_2,x_3)+l_1 g_1(x_1,x_2,x_3)+l_2 g_2(x_1,x_2,x_3)$$

<u>Ñieu kien ñu ve ñònh thöc Hess, xet 2 ñònh thöc:</u>

xet 1 ñònh thöc H_3: $H_3 < 0$ thì CĐ va $H_3 > 0$ thì CT

$$H_3=\left|\begin{array}{cc|ccc} 0 & 0 & f''_{l_1x_1} & f''_{l_1x_2} & f''_{l_1x_3} \\ 0 & 0 & f''_{l_2x_1} & f''_{l_2x_2} & f''_{l_2x_3} \\ \hline f''_{l_1x_1} & f''_{l_2x_1} & f''_{x_1x_1} & f''_{x_1x_2} & f''_{x_1x_3} \\ f''_{l_1x_2} & f''_{l_2x_2} & f''_{x_2x_1} & f''_{x_2x_2} & f''_{x_2x_3} \\ f''_{l_1x_3} & f''_{l_2x_3} & f''_{x_3x_1} & f''_{x_3x_2} & f''_{x_3x_3} \end{array}\right| = \left|\begin{array}{cc|ccc} 0 & 0 & g'_{1,x_1} & g'_{1,x_2} & g'_{1,x_3} \\ 0 & 0 & g'_{2,x_1} & g'_{2,x_2} & g'_{2,x_3} \\ \hline g'_{1,x_1} & g'_{2,x_1} & f''_{x_1x_1} & f''_{x_1x_2} & f''_{x_1x_3} \\ g'_{1,x_2} & g'_{2,x_2} & f''_{x_2x_1} & f''_{x_2x_2} & f''_{x_2x_3} \\ g'_{1,x_3} & g'_{2,x_3} & f''_{x_3x_1} & f''_{x_3x_2} & f''_{x_3x_3} \end{array}\right|$$

- Tööng töï ham 3 bien: L(x,y,z, λ) = f(x,y,z) + $\lambda\ \varphi$ (x,y,z)

  Thì $d^2L = L''_{x^2}\,dx^2 + L''_{y^2}\,dy^2 + L''_{z^2}\,dz^2 + 2L''_{xy}\,dxdy + 2L''_{xz}\,dxdz + 2L''_{yz}\,dydz$.

3.3 Cac ví du:

<u>*Ví du 1:*</u>

Tìm cac ñiem tôùi han cua: $f\left(x,y,z\right)= x+\frac{y^2}{4x}+\frac{z^2}{y}+\frac{2}{z}$, $x \neq 0$, $y \neq 0$, $z \neq 0$.

Dung ñieu kien cap 2 cua ñao ham ñe phan biet ñiem Max va Min.

Giaûi: Giaûi 3 phöông trình: $f_x^/ = 1 - \frac{y^2}{4x^2} = 0$; $f_y^/ = \frac{y}{2x} - \frac{z^2}{y^2} = 0$;

$f_z^/ = \frac{2z}{y} - \frac{2}{z^2} = 0 <=> \left\{ \frac{y}{2x} = \pm 1, \frac{y}{2x} = \frac{z^2}{y^2}, \frac{z}{y} = \frac{1}{z^2} \right\}$, vaäy y, x, z cuøng daáu.

Khử y trong 3 phương trình ñeå coøn phöông trình 2 bieán x, z: z^8=1==> $z = \pm 1$.....

tìm ñöôïc ñuùng 2 nghieäm döøng: M(1/2, 1, 1) vaø N(-1/2, -1, -1). Tính:

$$f_{xx}^{//} = \frac{y^2}{2x^3} \qquad f_{yy}^{//} = \frac{1}{2x} + \frac{2z^2}{y^3} \qquad f_{zz}^{//} = \frac{2}{y} + \frac{4}{z^3}$$

$$f_{xy}^{//} = -\frac{y}{2x^2}, \qquad f_{xz}^{//} = 0 \qquad f_{yz}^{//} = \frac{-2z}{y^2}$$

+ Baây giôø xeùt nghieäm taïi M(1/2, 1, 1):

Xeùt vi phaân baäc 2:

$$d^2 f = f_{xx}^{//}dx^2 + f_{yy}^{//}dy^2 + f_{zz}^{//}dz^2 + 2\left(f_{xy}^{//}dxdy + f_{xz}^{//}dxdz + f_{yz}^{//}dydz \right)$$

laø moät daïng toaøn phöông theo dx,dy. Ta coù ma traän:

$$\begin{vmatrix} f_{xx}^{//} & f_{xy}^{//} & f_{xz}^{//} \\ f_{yx}^{//} & f_{yy}^{//} & f_{yz}^{//} \\ f_{zx}^{//} & f_{zy}^{//} & f_{zz}^{//} \end{vmatrix} = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{vmatrix} = H_3 = 32 > 0, \ H_1 = 4 > 0,$$

$$H_2 = \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 8 > 0$$

$d^2f(M) > 0$ (DTP xaùc ñònh döông) $\forall dx, dy$ => M(1/2, 1, 1) cöïc tieåu.

+ Baây giôø xeùt nghieäm taïi N(-1/2, -1, -1)

$H_1 = -4 < 0$, $H_2 = \begin{vmatrix} -4 & 2 \\ 2 & -3 \end{vmatrix} = 8 > 0$

$$\begin{vmatrix} f_{xx}^{//} & f_{xy}^{//} & f_{xz}^{//} \\ f_{yx}^{//} & f_{yy}^{//} & f_{yz}^{//} \\ f_{zx}^{//} & f_{zy}^{//} & f_{zz}^{//} \end{vmatrix} = \begin{vmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -6 \end{vmatrix} = H_3 = -32 < 0,$$

Vaäy $d^2f(N) < 0$ (DTP xaùc ñònh aâm) $\forall dx, dy$

=> N(-1/2, -1, -1) cöïc ñaïi.

<u>*Caùch 2:*</u>

- Xeùt nghieäm taïi M(1/2, 1, 1):

Ta tính tröïc tieáp töø vi phaân baäc 2:

$$d^2 f = f_{xx}^{//}dx^2 + f_{yy}^{//}dy^2 + f_{zz}^{//}dz^2 + 2\left(f_{xy}^{//}dxdy + f_{xz}^{//}dxdz + f_{yz}^{//}dydz \right) \text{ laø:}$$

$d^2 f = 4dx^2 + 3dy^2 + 6dz^2 - 4dxdy - 4dydz$, sau moät soá tính toaùn :

$$d^2 f = 4\left(dx - \frac{dy}{2} \right)^2 + 2(dy - dz)^2 + 4dz^2 > 0 \text{ ==> M(1/2, 1, 1) CT}$$

- Xeùt nghieäm taïi N(-1/2, -1, -1):

Ta tính tröïc tieáp töø vi phaân baäc 2:

$d^2f = f''_{xx}dx^2 + f''_{yy}dy^2 + f''_{zz}dz^2 + 2\left(f''_{xy}dxdy + f''_{xz}dxdz + f''_{yz}dydz\right)$ la:

$d^2f = -4dx^2 - 3dy^2 - 6dz^2 + 4dxdy + 4dydz$, sau mot so tính toan :

$d^2f = -4\left(dx - \dfrac{dy}{2}\right)^2 - 2(dy - dz)^2 - 4dz^2 < 0$ =>N(-1/2, -1, -1) CÑ ¢

Ví du 2: Xet lai ví du tröôc ñay: Ham: z=f(x,y) = $x^3 + y^3 - 3xy$

- Tìm tap cac ñiem döng:

  $z'_x = 3x^2 - 3y = 0$

  $z'_y = 3y^2 - 3x = 0$

Giai he nay ta co 2 ñiem döng la $M_1(1,1)$ va $M_2(0,0)$

- Ta co: $z''_{x^2} = 6x$, $z''_{xy} = -3$, $z''_{y^2} = 6y$

  Va vi phan cap 2: $d^2f = f''_{xx}dx^2 + f''_{yy}dy^2 + 2f''_{xy}dxdy$

♣ Tai M_1: $d^2f = 6dx^2 + 6dy^2 - 6dxdy = 6\left(dy - \dfrac{dx}{2}\right)^2 + \dfrac{9dx^2}{2} > 0$,

nen M_1 la ñiem cöc tieu.

Co the thay bang $H_1 = 6>0$, $H_2 = 36 - 9 > 0$

♣ Tai M_2: $d^2f = -3dxdy = ?$ co the am, co the döông,

nen M_2 khong la ñiem cöc trò. Co the thay bang $H_2 = -9<0$. ¢

Ví du 3:

Tìm cac ñiem cöc trò cua: $f(x,y,z) = x + \dfrac{2y}{x} + \dfrac{z}{2y} + \dfrac{1}{z}$,

Giai: $D = R^3 - \{(0,0,0)\}$. Giai 3 phöông trình: $f'_x = 1 - \dfrac{2y}{x^2} = 0$; $f'_y = \dfrac{2}{x} - \dfrac{z}{2y^2} = 0$;

$f'_z = \dfrac{1}{2y} - \dfrac{1}{z^2} = 0 \iff \left\{x^2 = 2y, 4y^2 = xz, z^2 = 2y\right\}$,

Vay y> 0, x va z bang nhau va cung dau. Giai ra:

Tìm ñöôc ñung 2 nghiem döng: M(1, 1/2, 1) va N(-1, 1/2, -1).

Tính: $f''_{xx} = \dfrac{4y}{x^3}$ $f''_{yy} = \dfrac{z}{y^3}$ $f''_{zz} = \dfrac{2}{z^3}$

$f''_{xy} = -\dfrac{2}{x^2}$, $f''_{xz} = 0$ $f''_{yz} = \dfrac{-1}{2y^2}$

+ Bay giô xet nghiem tai M(1, 1/2, 1):

Xet vi phan bac 2:

$d^2f = f''_{xx}dx^2 + f''_{yy}dy^2 + f''_{zz}dz^2 + 2\left(f''_{xy}dxdy + f''_{xz}dxdz + f''_{yz}dydz\right)$

la mot dang toan phöông theo dx, dy, dz. Ta co ñònh thöc ma tran Hess:

$$\begin{vmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{vmatrix} = \begin{vmatrix} 2 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 2 \end{vmatrix} = H_3 = 16 > 0,\ H_1 = 2 > 0,$$

$$H_2 = \begin{vmatrix} 2 & -2 \\ -2 & 8 \end{vmatrix} = 12 > 0$$

$d^2f(M) > 0$ (DTP xaùc ñònh döông) $\forall dx, dy, dz$ => M(1, 1/2, 1) cöïc tieåu.

<u>Caùch khaùc:</u> $d^2 f = 2dx^2 + 8dy^2 + 2dz^2 - 4dxdy - 4dydz =$

$= 2(dx - dy)^2 + 4dy^2 + 2(dz - dy)^2 > 0$ => M(1, 1/2, 1) cöïc tieåu.

+ Baây giôø xeùt nghieäm taïi N(-1, 1/2, -1)

$H_1 = -2 < 0$, $H_2 = \begin{vmatrix} -2 & -2 \\ -2 & -8 \end{vmatrix} = 12 > 0$

$$\begin{vmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{vmatrix} = \begin{vmatrix} -2 & -2 & 0 \\ -2 & -8 & -2 \\ 0 & -2 & -2 \end{vmatrix} = H_3 = -16 < 0,$$

Vaäy $d^2f(N) < 0$ (DTP xaùc ñònh aâm) $\forall dx, dy, dz$

=> N(-1, -1/2, -1) cöïc ñaïi.

<u>Caùch khaùc:</u> $d^2 f = -2dx^2 - 8dy^2 - 2dz^2 - 4dxdy - 4dydz =$

$= -2(dx + dy)^2 - 4dy^2 - 2(dz + dy)^2 < 0$ => N(-1, -1/2, -1) cöïc ñaïi. ¢

<u>Ví duï 5:</u> Tìm cöïc trò cuûa f(x, y, z) =2x+y+3z.

Vôùi ñieàu kieän: g(x,y,z) = $x^2 + 4y^2 + 2z^2 - 35$ =0.

+ *<u>Caùch 1:</u>*

Haøm Lagrange: $L = f + l g = 2x + y + 3z + l\left(x^2 + 4y^2 + 2z^2 - 35\right)$

$L'_x = 2 + 2lx$, $L'_y = 1 + 8ly$, $L'_z = 3 + 4lz$, $L'_l = x^2 + 4y^2 + 2z^2 - 35$

$L''_{xx} = 2l$, $L''_{yy} = 8l$, $L''_{zz} = 4l$, $L''_{ll} = 0$

$L''_{xy} = 0$, $L''_{xz} = 0$, $L''_{yz} = 0$, $L''_{xl} = 2x$, $L''_{yl} = 8y$, $L''_{zl} = 4z$

$\overrightarrow{\nabla L} = \vec{0} \Leftrightarrow M_1\left(4, \frac{1}{2}, 3, l = \frac{-1}{4}\right)$, $M_2\left(-4, -\frac{1}{2}, -3, l = \frac{1}{4}\right)$

coù hai ñieåm döøng. Xeùt vi phaân baäc 2 .

$$d^2L = L''_{xx}dx^2 + L''_{yy}dy^2 + L''_{zz}dz^2 + 2\left(L''_{xy}dxdy + L''_{xz}dxdz + L''_{yz}dydz\right)$$

laø moät daïng toaøn phöông theo 3 bieán: dx,dy,dz. Chöùng minh ñieàu naøy töông töï nhö tröôøng hôïp 2 bieán.

- Xeùt vi phaân baäc 2 taïi $M_1\left(4, \frac{1}{2}, 3, l = \frac{-1}{4}\right)$, f=17.5

$$d^2L\Big|_{M_1} = L''_{xx}dx^2 + L''_{yy}dy^2 + L''_{zz}dz^2 = -\frac{1}{2}dx^2 - 2dy^2 - dz^2 < 0$$

Nen $M_1\left(4,\frac{1}{2},3,\, l=\frac{-1}{4}\right)$ la CÑ.

- Xet vi phan bac 2 tai $M_2\left(-4,-\frac{1}{2},-3,\, l=\frac{1}{4}\right)$, f= -17.5

$$d^2L\Big|_{M_2}=L''_{xx}dx^2+L''_{yy}dy^2+L''_{zz}dz^2=\frac{1}{2}dx^2+2dy^2+dz^2>0$$

Nen $M_2\left(-4,-\frac{1}{2},-3,\, l=\frac{1}{4}\right)$ la CT.

\+ *Cach 2:* Dung ñònh thöùc, xet 2 ñònh thöùc (n=3,m=1).

Tai $M_1\left(4,\frac{1}{2},3,\, l=\frac{-1}{4}\right)$, $(-1)^k H_k>0$, $k=\overline{m+1,n}=\overline{2,3}$, that vay:

Ta xet: $H_2=\begin{vmatrix}0 & g'_x & g'_y\\ g'_x & L''_{xx} & L''_{xy}\\ g'_y & L''_{yx} & L''_{yy}\end{vmatrix}=\begin{vmatrix}0 & 8 & 4\\ 8 & \frac{-1}{2} & 0\\ 4 & 0 & -2\end{vmatrix}$ =136 > 0

$H_3=\begin{vmatrix}0 & g'_x & g'_y & g'_z\\ g'_x & L''_{xx} & L''_{xy} & L''_{xz}\\ g'_y & L''_{yx} & L''_{yy} & L''_{yz}\\ g'_z & L''_{zx} & L''_{zy} & L''_{zz}\end{vmatrix}=\begin{vmatrix}0 & 8 & 4 & 12\\ 8 & \frac{-1}{2} & 0 & 0\\ 4 & 0 & -2 & 0\\ 12 & 0 & 0 & -1\end{vmatrix}$ = -280 <0, Tai M_1 CÑ.

Töông töï tai $M_2\left(-4,-\frac{1}{2},-3,\, l=\frac{1}{4}\right)$, CT. That vay:

Ta xet: $H_2=\begin{vmatrix}0 & g'_x & g'_y\\ g'_x & L''_{xx} & L''_{xy}\\ g'_y & L''_{yx} & L''_{yy}\end{vmatrix}=\begin{vmatrix}0 & -8 & -4\\ -8 & \frac{1}{2} & 0\\ -4 & 0 & 2\end{vmatrix}$ = -136 <0

$H_3=\begin{vmatrix}0 & g'_x & g'_y & g'_z\\ g'_x & L''_{xx} & L''_{xy} & L''_{xz}\\ g'_y & L''_{yx} & L''_{yy} & L''_{yz}\\ g'_z & L''_{zx} & L''_{zy} & L''_{zz}\end{vmatrix}=\begin{vmatrix}0 & -8 & -4 & -12\\ -8 & \frac{1}{2} & 0 & 0\\ -4 & 0 & 2 & 0\\ -12 & 0 & 0 & 1\end{vmatrix}$ = -280<0, tai M_2 CT. ¢

Ví du 6: Lay ví du tröôùc z = xy, thoa $\varphi(x,y) = (x-1)^2 + y^2 - 1 = 0$

$L = z + \lambda.\varphi = xy + \lambda\,[(x-1)^2 + y^2 - 1]$

Ta xet 2 ñiem döøng $M_1\left(\frac{3}{2},\frac{\sqrt{3}}{2},\, l=\frac{-\sqrt{3}}{2}\right)$ va $M_2\left(\frac{3}{2},\frac{-\sqrt{3}}{2},\, l=\frac{\sqrt{3}}{2}\right)$.

$L''_{x^2}=2l=L''_{y^2}$, $L''_{xy}=1$, $L''_{xl}=2(x-1)=j'_x$, $L''_{yl}=2y=j'_y$

•Tai $M_1\left(\frac{3}{2},\frac{\sqrt{3}}{2},\, l=\frac{-\sqrt{3}}{2}\right)$, z= 1.299, Ta xet: $H_2 = \begin{vmatrix} 0 & \varphi'_x & \varphi'_y \\ \varphi'_x & a_{11} & a_{12} \\ \varphi'_y & a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & \sqrt{3} \\ 1 & -\sqrt{3} & 1 \\ \sqrt{3} & 1 & -\sqrt{3} \end{vmatrix} = 6\sqrt{3}$ >0, M_1 la CÑ

•Tai $M_2(\frac{3}{2}, \frac{-\sqrt{3}}{2}, \lambda = \frac{\sqrt{3}}{2})$, z= -1.299,

Ta xet: $H_2 = \begin{vmatrix} 0 & \varphi'_x & \varphi'_y \\ \varphi'_x & a_{11} & a_{12} \\ \varphi'_y & a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & -\sqrt{3} \\ 1 & \sqrt{3} & 1 \\ -\sqrt{3} & 1 & \sqrt{3} \end{vmatrix} = -6\sqrt{3}$ >0, M_2 la CT

•Tai $M_0(0, 0, \lambda$ bat ky), Ta xet:

$$H_2 = \begin{vmatrix} 0 & \varphi'_x & \varphi'_y \\ \varphi'_x & a_{11} & a_{12} \\ \varphi'_y & a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & -2 & 0 \\ -2 & 2\lambda & 1 \\ 0 & 1 & 2\lambda \end{vmatrix} = -8\lambda$$, M_0 chöa co ket luan ¢

Ví du 7: Lay ví du tröôc z = xy, thoa φ(x,y) = 3x+2y – 5 = 0

L = f + λ. φ = xy + λ [3x+2y – 5]

Co 1 ñiem döng duy nhat $M(\frac{5}{6}, \frac{5}{4}, \lambda = \frac{-5}{12})$.

$L''_{x^2} = 0 = L''_{y^2}$, $L''_{xy} = 1$, $L''_{x\lambda} = 3 = \varphi'_x$, $L''_{y\lambda} = 2 = \varphi'_y$.

Tai $M(\frac{5}{6}, \frac{5}{4}, \lambda = \frac{-5}{12})$, z=25/24,

Ta xet: $H_2 = \begin{vmatrix} 0 & \varphi'_x & \varphi'_y \\ \varphi'_x & a_{11} & a_{12} \\ \varphi'_y & a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 0 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 12$ >0, M la CÑ ¢

Ví du 13: Tìm cöc trò ham $f(x_1, x_2) = 3x_1 - 4x_2$ vôi rang buoc

$\varphi(x_1, x_2) = 8x_1x_2 + 16x_1 - 8x_2^2 + 24x_2 + 104 = 0 \rightarrow$ hyperbol xien

Ham L: $L = f(x_1, x_2) + \lambda\varphi(x_1, x_2) =$

$= 3x_1 - 4x_2 + \lambda(8x_1x_2 + 16x_1 - 8x_2^2 + 24x_2 + 104)$, ñao ham:

$\begin{cases} L'_{x_1} = 3 + \lambda(8x_2 + 16) = 0 \\ L'_{x_2} = -4 + \lambda(8x_1 - 16x_2 + 24) = 0 \end{cases}$, khöû λ ô hai phöông trình nay:

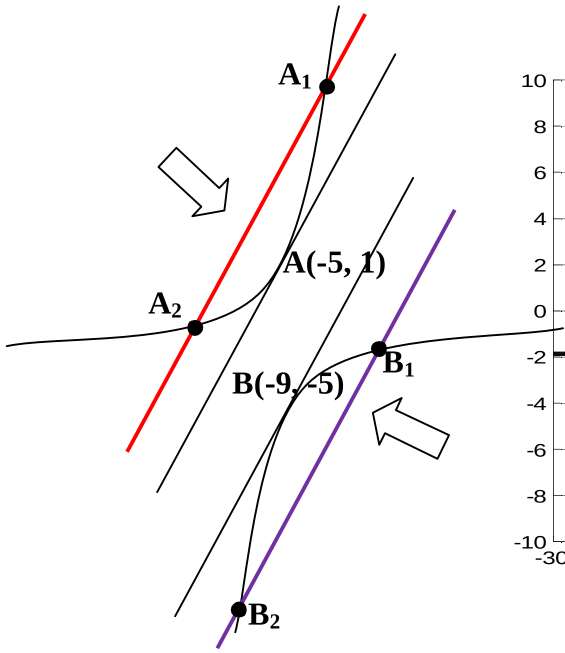
$\Rightarrow \frac{8x_2 + 16}{8x_1 - 16x_2 + 24} = \frac{x_2 + 2}{x_1 - 2x_2 + 3} = \frac{-3}{4}$ ==> $3x_1 - 2x_2 = -17$, giai he

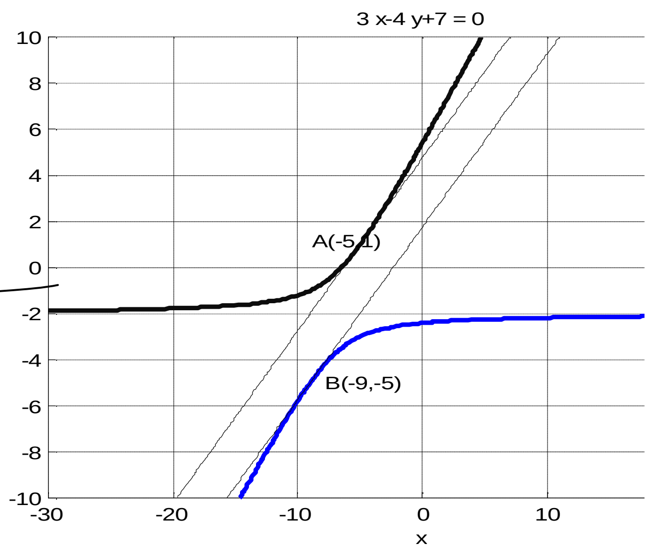
$\begin{cases} \varphi(x_1,x_2) = 8x_1x_2 + 16x_1 - 8x_2^2 + 24x_2 + 104 = 0 \\ 3x_1 - 2x_2 = -17 \end{cases}$, ta tìm ñöôc 2 ñap an cho boi ba (x_1, x_2, l) la:

A(-5, 1, -1/8) va B(-9, -5, 1/8)

•Tai A(-5, 1, λ= -1/8), Nhanh tren

$$H_2 = \begin{vmatrix} 0 & \varphi'_{x_1} & \varphi'_{x_2} \\ \varphi'_{x_1} & L''_{x_1x_1} & L''_{x_1x_2} \\ \varphi'_{x_2} & L''_{x_2x_1} & L''_{x_2x_2} \end{vmatrix} = \begin{vmatrix} 0, & 24, & -32 \\ 24, & 0, & -1 \\ -32, & -1, & 2 \end{vmatrix} = 384 \text{ --> CÑ}$$





$f(x_1, x_2) = 3x_1 - 4x_2 =$ -19. Va ñöông thang: $3x_1 - 4x_2 = -19$ la ñöông thang (töi mien cat)ñi töi tren ñi xuong (theo chieu tang mui ten cua C, $3x_1 - 4x_2 = C$) ñe tiep xuc CĐ vao nhanh tren hyperbol tai A(-5, 1).

Co the xet dạng vi phan d^2L: Töø $\varphi'_{x_1}dx_1 + \varphi'_{x_2}dx_2 = 0 \Leftrightarrow$

$24dx_1 - 32dx_2 = 0 \Leftrightarrow dx_1 = \frac{4}{3}dx_2$. Vay:

$d^2L = L''_{x_1^2}dx_1^2 + L''_{x_2^2}dx_2^2 + 2L''_{x_1x_2}dx_1dx_2 =$

$= 2dx_2^2 - 2dx_1dx_2 = 2dx_2^2 - 2\frac{4}{3}dx_2^2 < 0$ -->CÑ

•Tai B(-9, -5, λ=1/8), Nhanh döôi

$$H_2 = \begin{vmatrix} 0 & \varphi'_{x_1} & \varphi'_{x_2} \\ \varphi'_{x_1} & L''_{x_1x_1} & L''_{x_1x_2} \\ \varphi'_{x_2} & L''_{x_2x_1} & L''_{x_2x_2} \end{vmatrix} = \begin{vmatrix} 0, & -24, & 32 \\ -24, & 0, & 1 \\ 32, & 1, & -2 \end{vmatrix} = -384 \text{ --> CT}$$

$f(x_1, x_2) = 3x_1 - 4x_2 = -7$. Va ñöông thang: $3x_1 - 4x_2 = -7$

La ñöông thang (tö mien cat) ñi tö döôi ñi len xuong (theo chieu tang ngöôc mui ten cua C, $3x_1 - 4x_2 = C$) tiep xuc CT vao nhanh döôi hyperbol tai B(-9, -5).

Co the xet dạng vi phan d^2L:

Tö $\varphi'_{x_1}dx_1 + \varphi'_{x_2}dx_2 = 0 \Leftrightarrow -24dx_1 + 32dx_2 = 0 \Leftrightarrow dx_1 = \frac{4}{3}dx_2$

$d^2L = L''_{x_1^2}dx_1^2 + L''_{x_2^2}dx_2^2 + 2L''_{x_1x_2}dx_1dx_2 =$

$= -2dx_2^2 + 2dx_1dx_2 = -2dx_2^2 + 2\frac{4}{3}dx_2^2 > 0$ -->CT

Chu y:

0/ Nhanh tren co $f(A_1) = f(A_2)$ va nhanh döôi co $f(B_1) = f(B_2)$

1/ Nhanh tren co: $\lambda = \frac{\vec{\nabla} f}{\vec{\nabla} \varphi} = -\frac{1}{8}$, ngöôc vôi nhanh döôi co: $\lambda = \frac{\vec{\nabla} f}{\vec{\nabla} \varphi} = \frac{1}{8}$

2/ Ñöng ñan ño co mau thuan ve gia trò Max= -19 va Min= -7, tren 2 nhanh khac nhau, that ra ñay la Max, Min tren hai nhanh khac nhau, hay xem gia trò ham $f(x_1, x_2) = 3x_1 - 4x_2$ tren hai nhanh döôi ñay: That vay, ta se tính gia trò cua $f(x_1, x_2) = 3x_1 - 4x_2$ tren cac ñiem lan can cua ñiem cöc trò tren ca hai nhanh bang excel sau ñay:

Đồ thị nhanh tren: $f_1 = \frac{1}{2}\left(3 + x + \sqrt{x^2 + 14x + 61}\right)$,

Đồ thị nhanh döôi: $f_2 = \frac{1}{2}\left(3 + x - \sqrt{x^2 + 14x + 61}\right)$, à do tìm x_2, qua x_1, tö phương trình:

$\varphi(x_1, x_2) = 8x_1x_2 + 16x_1 - 8x_2^2 + 24x_2 + 104 = 0$,

$x_2 = \frac{1}{2}\left(3 + x_1 + \sqrt{x_1^2 + 14x_1 + 61}\right)$, $x_2 = \frac{1}{2}\left(3 + x_1 - \sqrt{x_1^2 + 14x_1 + 61}\right)$

	Nhanh tren	f1 -->max
x	y=f1(x)	f=3x-4y=C
-8	-0.69722	-21.2111
-7	-0.26795	-19.9282
-6	0.302776	-19.2111
-5	1	-19
-4	1.791288	-19.1652
-3	2.645751	-19.583
-2	3.541381	-20.1655

	Nhanh döôi	f2 -->min
x	y=f2(x)	f=3x-4y=C
-12	-7.54138	-5.83447
-11	-6.64575	-6.41699
-10	-5.79129	-6.83485
-9	-5	-7
-8	-4.30278	-6.7889
-7	-3.73205	-6.0718
-6	-3.30278	-4.7889

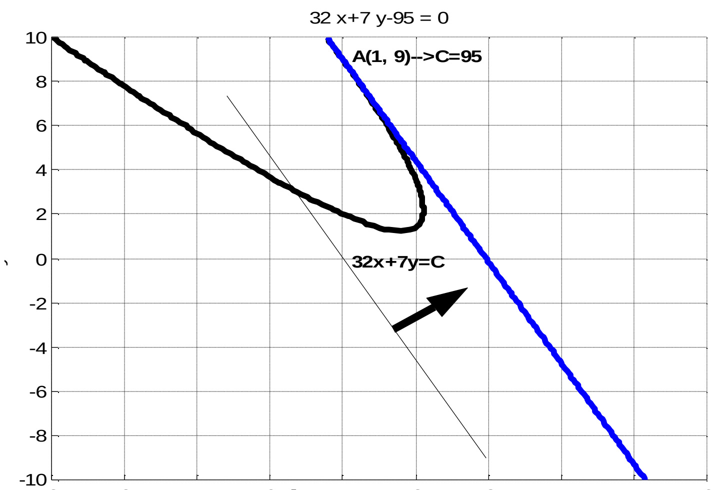
Ví du 14: Cho ham $f(x_1, x_2) = 32x_1 + 7x_2$, vôi rang buoc

$\varphi(x_1, x_2) = -9x_1^2 - 6x_1x_2 - x_2^2 + 40x_1 + 17x_2 - 49 = 0$ (parabol xien). Tìm cac cac trò va cac gia trò ham töông öng.

$$L = f + \lambda\varphi,\ (*)\begin{cases}(1): L'_{x_1} = 32 + \lambda(-18x_1 - 6x_2 + 40) = 0\\ (2): L'_{x_2} = 7 + \lambda(-6x_1 - 2x_2 + 17) = 0\\ (3): L'_{\lambda} = -9x_1^2 - 6x_1x_2 - x_2^2 + 40x_1 + 17x_2 - 49 = 0\end{cases}$$, chuyeån veá ôû (1) vaø (2) (ta

coù $\lambda \neq 0$), chia veá vôùi veá, ñeå khöû λ, ñieàu naøy ta laøm ñöôïc vì maãu soá coù theå giaû söû khaùc 0, do ta coù theå giaû söû: $\left(\vec{\nabla f}, \vec{\nabla \varphi}\right) \neq \left(\vec{0}, \vec{0}\right)$. Vaäy khử $\lambda \neq 0$ từ (1) vaø (2) → $-3x1 - x2 + 12$ =0, rồi thế vaøo (3) ta coù →

$-11x_1 + 11 = 0 \Rightarrow x_1 = 1$, theá vaøo (3) ta coù phöông trình baäc 2: $x_2^2 - 11x_2 = -18 = 0 \Rightarrow x_2 = 2,9$.



•Laáy $x_2 = 9$, $x_1 = 1$, theá vaøo (1) vaø (2) → λ=1,

vaäy coù nghieäm M$\left(x_1 = 1, x_2 = 9, \lambda = 1\right)$, → $f(1,9) = 32x_1 + 7x_2 = 95$

Vaø Ta xeùt: $H_2 = \begin{vmatrix} 0 & \varphi'_{x_1} & \varphi'_{x_2} \\ \varphi'_{x_1} & L''_{x_1x_1} & L''_{x_1x_2} \\ \varphi'_{x_2} & L''_{x_2x_1} & L''_{x_2x_2}\end{vmatrix} = \begin{vmatrix} 0 & -32 & -7 \\ -32 & -18 & -6 \\ -7 & -6 & -2\end{vmatrix} = 242$ >0, M laø CÑ.

•Laáy $x_2 = 2$, $x_1 = 1$ (→ khoâng thoûa $-3x1 - x2 + 12$ =0), hay theá vaøo (1) → 32+10λ=0 → λ= -32/10, vaø theá λ vaøo (2) thì khoâng thoûa, vì λ= -1, neân heä (*) voâ nghieäm.

Vaäy coù duy nhaát moät nghieäm maø thoâi: $\left(x_1 = 1, x_2 = 9, \lambda = 1\right)$ thoûa $\vec{\nabla L} = \vec{0}$, coøn nghieäm kia ($x_2 = 2$, $x_1 = 1$) khoâng thoûa $\vec{\nabla L} = \vec{0}$.

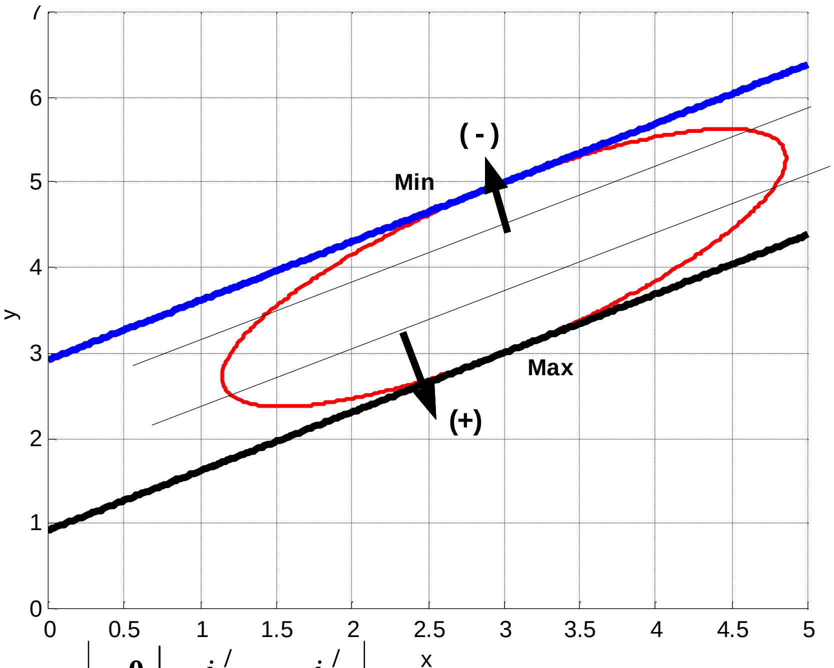
Ví duï 15: Cho haøm $f\left(x_1, x_2\right) = 9x_1 - 13x_2$, vôùi raøng buoäc

$\varphi\left(x_1, x_2\right) = 10x_1^2 - 18x_1x_2 + 13x_2^2 + 12x_1 - 50x_2 + 69 = 0$ (ellip xieân). Tìm caùc caùc trò vaø caùc giaù trò haøm töông öùng.

$$L = f + \lambda\varphi,\ (*)\begin{cases}(1): L'_{x_1} = 9 + \lambda(20x_1 - 18x_2 + 12) = 0\\ (2): L'_{x_2} = -13 + \lambda(-18x_1 + 26x_2 - 50) = 0\\ (3): L'_{\lambda} = 10x_1^2 - 18x_1x_2 + 13x_2^2 + 12x_1 - 50x_2 + 69 = 0\end{cases}$$,

Coù 2 ñieåm döøng:A($x_1 = 3$, $x_2 = 3$, λ=1/2), B($x_1 = 3$, $x_2 = 5$, λ=-1/2)

• Xeùt A($x_1 = 3$, $x_2 = 5$, λ=1/2). Vaø Ta xeùt: $f\left(x_1, x_2\right) = -38$



$$H_2 = \left|\begin{array}{c|cc} 0 & j'_{x_1} & j'_{x_2} \\ \hline j'_{x_1} & L''_{x_1x_1} & L''_{x_1x_2} \\ j'_{x_2} & L''_{x_2x_1} & L''_{x_2x_2} \end{array}\right| = \left|\begin{array}{c|cc} 0 & -18 & 26 \\ \hline -18 & 10 & -9 \\ 26 & -9 & 13 \end{array}\right| = -2548$$ <0, à A laø CT.

• Xeùt B($x_1 = 3$, $x_2 = 3$, λ= -1/2). Vaø Ta xeùt: $f\left(x_1, x_2\right) = -12$

$$H_2 = \left|\begin{array}{c|cc} 0 & j'_{x_1} & j'_{x_2} \\ \hline j'_{x_1} & L''_{x_1x_1} & L''_{x_1x_2} \\ j'_{x_2} & L''_{x_2x_1} & L''_{x_2x_2} \end{array}\right| = \left|\begin{array}{c|cc} 0 & 18 & -26 \\ \hline 18 & -10 & 9 \\ -26 & 9 & -13 \end{array}\right| = 2548$$ >0, à B laø CÑ.

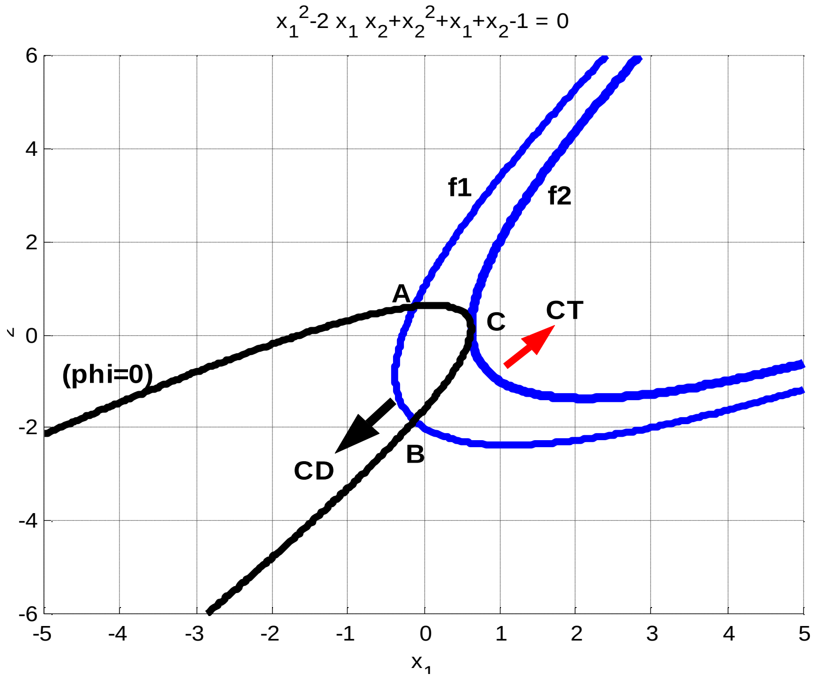
<u>Ví duï 16:</u> Cho haøm $f\left(x_1, x_2\right) = -7x_1 + x_2 + 7 + x_1^2 - 2x_1x_2 + x_2^2$, vôùi raøng buoäc: $j\left(x_1, x_2\right) = x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2 - 1 = 0$ (2 parabol xieân). Tìm caùc caùc trò vaø caùc giaù trò haøm töông öùng.

$$L = f + l\,j\,,\ (*)\begin{cases} (1): L'_{x_1} = -7 + 2x_1 - 2x_2 + l\left(2x_1 - 2x_2 + 1\right) = 0 \\ (2): L'_{x_2} = 1 - 2x_1 + 2x_2 + l\left(-2x_1 + 2x_2 + 1\right) = 0, \\ (3): L'_{l} = x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2 - 1 = 0 \end{cases}$$

Giaûi (1) vaø (2) : ñaët u= $2x_1 - 2x_2$ à khöû λ ta coù u=1 à $2x_1 - 2x_2$ =1 theá vaøo (3) giaûi ra ta coù ñieåm tieáp xuùc C(5/8, 1/8, λ=3), f(5/8, 1/8) =3 à min.
ta xeùt ñònh thöùc :

$$H_2 = \begin{vmatrix} 0 & \varphi'_{x_1} & \varphi'_{x_2} \\ \varphi'_{x_1} & L''_{x_1x_1} & L''_{x_1x_2} \\ \varphi'_{x_2} & L''_{x_2x_1} & L''_{x_2x_2} \end{vmatrix} = \begin{vmatrix} 0 & 2 & 0 \\ 2 & 8 & -8 \\ 0 & -8 & 8 \end{vmatrix} = -32$$ <0, à C laø CT.

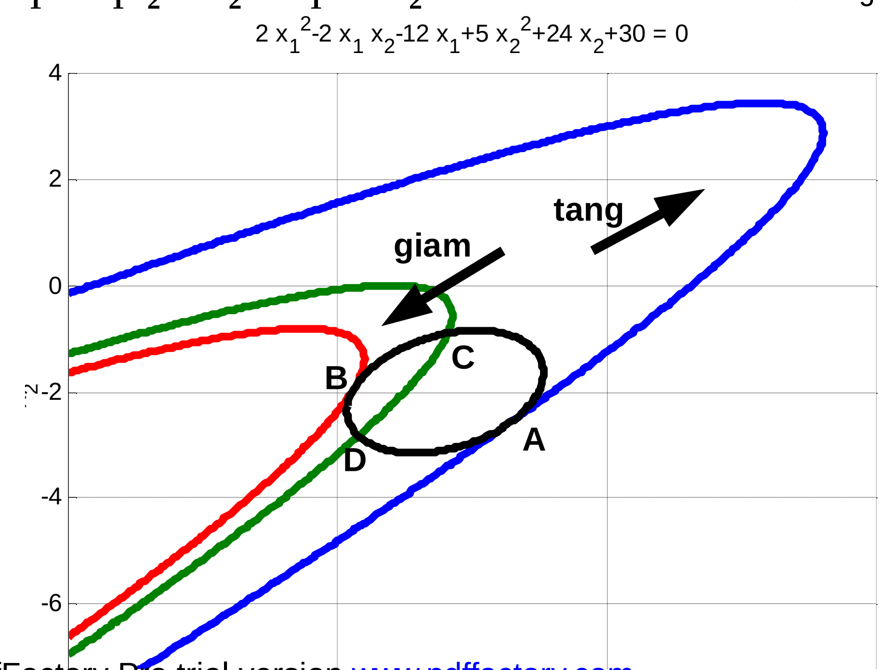
Nghieäm laïi treân hình veõ: ta coù $f=f_1=9$, $f=f_2=3$, giaù trò f caøng lôùn khi ñænh cuûa f caøng ñi xuoáng phía $-\infty$ vaø luoân coù $f_1(A)=f_1(B)$, chieàu



ngöôïc laïi f (laøm f giaûm) ñaït min taïi C(5/8, 1/8, λ=3) à khi 2 parabol naøy tieáp xuùc taïi C.
Xeùt theâm moät ví duï nöõa, nhöõng keát quaû laø nhôø coâng cuï cuûa toaùn chuyeân ñeà phöông phaùp tính, ñeå giaûi caùc phöông trình phi tuyeán.

Ví duï 17: Cho haøm $f(x_1, x_2) = x_1^2 + 13x_2 - 2x_1 - 4x_1x_2 + 4x_2^2 + 1$, vôùi raøng buoäc:

$\varphi(x_1, x_2) = 2x_1^2 - 2x_1x_2 + 5x_2^2 - 2x_1 + 24x_2 + 30 = 0$. Tìm caùc caùc trò vaø caùc giaù trò haøm töông öùng.



Kết quả: Ñænh parabol tăng +∞ theo chiều mũi teân, laøm giaùtrị f tăng, như hình vẽ, hai ñieåm tieáp xuùc taïi elip $j\left(x_1, x_2\right)$=0 coù ñònh laøm $f\left(x_1, x_2\right)$ ñaït cöïc trò laø: (Hoïc phaàn PPT seõ giaûi quyeát vaán ñeà naøy)

A(x_1=3.195188210, x_2= -2.589062020) à f_{max}= 31.06417536
B(x_1=0.3023821985, x_2= -1.936393601) à f_{min}= -7.34584169
Hai giaù trị tại 2 iểm C, D baèng nhau: f(C)=f(D)= 0.

BAØI 4: MOÄT SOÁ BAØI TOAÙN CÖÏC TRÒ TRONG KINH TEÁ

4.1 Baøi toaùn cho xí nghieäp saûn xuaát nhieàu saûn phaåm trong ñieàu kieän caïnh tranh hoaøn haûo:

Xeùt moät xí nghieäp saûn xuaát n =2 loaïi saûn phaåm trong ñieàu kieän caïnh tranh hoaøn haûo (nhaø saûn xuaát phaûi baùn saûn phaåm vôùi giaù do thò tröôøng quyeát ñònh). Bieát giaù (Price) cuûa caùc saûn phaåm treân laø P_1; P_2 vaø haøm toång chi phí (Cost) xeùt trong moät ñôn vò thôøi gian laø C =C (Q_1,Q_2), laø haøm tính theo naêng löôïng tieâu hao, treân hai möùc saûn löôïng (Quantity) Q_1, Q_2. Neân noù phaûi laø baäc 2 töông hoã cuûa Q_1, Q_2. Toång quaùt haøm chi phí laø haøm:

$C(Q)=C(Q_1,Q_2,..,Q_n)=Q^T AQ+BQ+const$, trong ñoù:

$B,Q^T=[Q_1 \quad Q_2 \quad ... \quad Q_n]\in M_{1\times n}$, $A\in M_{n\times n}$, $const\in R$, goàm caùc yeáu toá chi phí lieân quan ñeán naêng löôïng thì thuoäc daïng toaøn phöông $Q^T AQ$, vì toác ñoä thay ñoåi cuûa $C(Q)$ (töùc ñaïo haøm cuûa noù) laïi phuï thuoäc tröïc tieáp vaøo bieán Q. Ngoaøi ra coøn coù caùc yeáu toá chæ tæ leä tuyeán tính BQ theo saûn löôïng , vaø theâm haèng soá $Const$ laø chi phí ban ñaàu boû ra. Vaø nhö vaäy chi phí laø moät loaïi quadric toång quaùt trong R^n.

Baøi toaùn laø tìm möùc saûn löôïng Q_1, Q_2 cuûa caùc loaïi saûn phaåm treân trong moät ñôn vò thôøi gian ñeå xí nghieäp coù lôïi nhuaän toái ña.

Giaû söû Q_1, Q_2 laø saûn löôïng cuûa caùc loaïi saûn phaåm ñöôïc saûn xuaát trong moät ñôn vò thôøi gian. Khi ñoù doanh thu (Revenue) seõ laø:

$R=P_1Q_1 + P_2Q_2$.

Lôïi nhuaän (Profit) thu ñöôïc: π = R – C.

$\pi = P_1Q_1 + P_2Q_2 – C(Q_1,Q_2)$.

Möùc saûn löôïng phaûi tìm laø giaù trò Q_1, Q_2 döông ñeå laøm π ñaït cöïc ñaïi.

Ví duï: Giaû söû moät xí nghieäp saûn xuaát hai loaïi saûn phaåm vôùi giaù
P_1 = 10; P_2 = 10. Haøm toång chi phí

$C = C(Q_1,Q_2) = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$.

Tìm möùc saûn löôïng Q_1, Q_2 ñeå xí nghieäp coù lôïi nhuaän toái ña.
Doanh thu cuûa xí nghieäp laø: $R= P_1Q_1 + P_2Q_2 = 10Q_1 + 10Q_2$.
Lôïi nhuaän cuûa xí nghieäp: π = R – C=

$\pi= P_1Q_1 + P_2Q_2 – 2Q_1^2 – Q_1Q_2 – 2 Q_2^2$.

Vaäy $\pi =10Q_1 + 10Q_2 – 2Q_1^2 – Q_1Q_2 – 2 Q_2^2$.
Ta tìm cöïc trò cuûa π.

$$\begin{cases}\dfrac{\partial\pi}{\partial Q_1}=0\\[2mm] \dfrac{\partial\pi}{\partial Q_2}=0\end{cases} \Leftrightarrow \begin{cases}P_1-4Q_1-Q_2=0\\ P_2-4Q_2-Q_1=0\end{cases} \Leftrightarrow \begin{cases}4Q_1+Q_2=P_1\\ Q_1+4Q_2=P_2\end{cases}$$

$$\Leftrightarrow Q_1=\frac{4P_1-P_2}{15}=2\ ;\ Q_2=\frac{4P_2-P_1}{15}=2,\quad \frac{\partial^2\pi}{\partial^2 Q_1}=-4;\quad \frac{\partial^2\pi}{\partial Q_1\partial Q_2}=-1\ ;\quad \frac{\partial^2\pi}{\partial^2 Q_2}=-4,$$

Vaäy ta coù A = -4 < 0 vaø $\Delta = AC - B^2 = 16-1 = 15 > 0$.

Suy ra π ñaït cöïc ñaïi taïi ñieåm $Q_1 = 2$, $Q_2 = 2$. Vaäy möùc saûn löôïng maø xí nghieäp saûn xuaát ñeå coù lôïi nhuaän toái ña laø $Q_1 = 2$ vaø $Q_2 = 2$.

<u>Caùch khaùc:</u> Tính vi phaân baäc 2: $d^2p = -4dQ_1^2 - 4dQ_2^2 + 2(-1)dQ_1dQ_2 =$

$$= -4\left(dQ_1 + \frac{dQ_2}{4}\right)^2 - \frac{15}{4}dQ_2^2 < 0$$ ==> ($Q_1 = 2$ vaø $Q_2 = 2$) CÑ.

4.2 Baøi toaùn cho xí nghieäp saûn xuaát nhieàu saûn phaåm trong ñieàu kieän saûn xuaát ñoäc quyeàn.

Giaû söû moät xí nghieäp saûn xuaát ñoäc quyeàn n=2 loaïi saûn phaåm. Bieát haøm caàu cuûa xí nghieäp veà 2 loaïi saûn phaåm treân moät ñôn vò thôøi gian laø:

$Q_{D1} = D_1(p_1,p_2)$

$Q_{D2} = D_2(p_1,p_2)$ (p_1, p_2 laø giaù baùn)

Thoâng thöôøng $i=\overline{1,2},\ \ p_i\uparrow\ \Rightarrow Q_{D_i}\downarrow$ hay ngöôïc laïi.

Vaø haøm toång chi phí cuûa xí nghieäp: C =C (Q_1,Q_2)

Tìm möùc saûn löôïng Q_1,Q_2 ñeå xí nghieäp coù lôïi nhuaän toái ña.

Giaû söû Q_1,Q_2 laø saûn löôïng cuûa caùc loaïi saûn phaåm maø xí nghieäp saûn xuaát trong moät ñôn vò thôøi gian. Ñeå baùn heát caùc loaïi saûn phaåm treân xí nghieäp phaûi baùn vôùi giaù p_1,p_2 sao cho:

$Q_{D1} = Q_1 = D_1(p_1,p_2)$

$Q_{D2} = Q_2 = D_2(p_1, p_2)$ (Do quy luaät caàu)

Töø ñaáy tìm ra ñöôïc p_1;p_2 theo Q_1; Q_2. khi ñoù doanh thu cuûa xí nghieäp laø:

$R = p_1Q_1+p_2Q_2$

Lôïi nhuaän (Profit) cuûa xí nghieäp laø: $\pi = R - C$,

möùc saûn löôïng tìm ra ñöôïc Q_1,Q_2 phaûi tìm laø giaù trò Q_1,Q_2 döông sao cho p ñaït cöïc ñaïi.

<u>*Ví duï:*</u>

Laáy $Q_{D1} = 40 - 2p_1 + p_2$; $Q_{D2} = 15 + p_1 - p_2$ vaø haøm toång chi phí trong moät ñôn vò thôøi gian laø: $C = C(Q_1,Q_2) = Q_1^2 + Q_1Q_2 + Q_2^2$. tìm möùc saûn löôïng ñeå xí nghieäp coù lôïi nhuaän toái ña.

Lyù luaän nhö treân ta ñi tôùi heä phöông trình.

$Q_{D1} = 40 - 2p_1 + p_2$

$Q_{D2} = 15 + p_1 - p_2$

Giaûi heä tìm ñöôïc:

$p_1 = 55 - Q_1 - Q_2$; $p_2 = 70 - Q_1 - 2Q_2$

Doanh thu cuûa xí nghieäp laø:

$R = p_1Q_1 + p_2Q_2 = (55 - Q_1 - Q_2)\,Q_1 + (70 - Q_1 - 2Q_2)\,Q_2$

Lôïi nhuaän (Profit) cuûa xí nghieäp laø: $\pi = R - C =$

$=(55 - Q_1 - Q_2)\,Q_1 + (70 - Q_1 - 2Q_2)\,Q_2 - (Q_1^2 + Q_1Q_2 + Q_2^2) =$

$= -2Q_1^2 - 3Q_2^2 - 3Q_1Q_2 + 55\mathbf{Q_1} + 70Q_2$

Ta tìm ñöôc cöc trò cua π:

$$\begin{cases} \dfrac{\partial \pi}{\partial \mathbf{Q_1}} = \mathbf{0} \\ \dfrac{\partial \pi}{\partial \mathbf{Q_2}} = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \mathbf{-4Q_1 - 3Q_2 + 55 = 0} \\ \mathbf{-6Q_2 - 3Q_1 + 70 = 0} \end{cases} \Leftrightarrow \begin{cases} \mathbf{4Q_1 + 3Q_2 = 55} \\ \mathbf{3Q_1 + 6Q_2 = 70} \end{cases}$$

$\Leftrightarrow \begin{cases} \mathbf{Q_1 = 8} \\ \mathbf{Q_2 = \dfrac{23}{3}} \end{cases}$, va co: $\dfrac{\partial^2 \pi}{\partial \mathbf{Q_1}^2} = \mathbf{-4}$; $\dfrac{\partial^2 \pi}{\partial \mathbf{Q_2}^2} = \mathbf{-6}$; $\dfrac{\partial^2 \pi}{\partial \mathbf{Q_1} \partial \mathbf{Q_2}} = \mathbf{-3}$.

Vay A = -4; B = -3; C = -6

$\begin{cases} \mathbf{A = -4 < 0} \\ \Delta = \mathbf{AC\text{-}B^2 = 24\text{-}9 = 15 > 0} \end{cases}$ $\Rightarrow$ π ñat cöc ñai tai (Q_1,Q_2) = (8,23/3).

Vay möc san löông (Q_1,Q_2) ma xí nghiep san xuat ñe co lôi nhuan toi ña la (8,23/3).

<u>*Cach khac:*</u> Tính vi phan bac 2: $d^2p = -4dQ_1^2 - 6dQ_2^2 + 2(-3)dQ_1 dQ_2 =$

$$= -6\left(dQ_2 + \frac{dQ_1}{2}\right)^2 - \frac{5}{2}dQ_1^2 < 0$$ ==> (Q_1,Q_2) = (8,23/3) CÑ.

4.3 Bai toan ngöôi tieu dung:

Gia sö ngöôi tieu dung ñònh dung mot so tien la B ñe mua sam n=2 loai hang. Biet rang gia cua cac loai hang la $\mathbf{p_1;p_2}$,ham höu dung cho n=2 loai hang tren: u=u(x,y), trong ño x,y la so löông hang thö nhat thö hai töông öng ma ngöôi ño mua. Hay xac ñònh so löông hang ma ngöôi tieu dung se mua, sao cho gia trò sö dung lôn nhat.

Ta phai tìm cöc trò ham so u=u(x,y) vôi ñieu kien: $p_1x + p_2y$ =B.

Ñay la bai toan tìm cöc trò co ñieu kien.

<u>*Ví du:*</u> $\mathbf{u(x,y) = \sqrt{(x+2)(y+1)}}$ va p_1=4; p_2=6; B=200

Ta tìm cöc trò cua ham $\mathbf{u = \sqrt{(x+2)(y+1)}}$ vôi ñieu kien:

φ(x,y)=4x + 6y – 200=0

Tìm ñiem döng (x_0,y_0) va so λ thoa he phöông trình:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial \mathbf{x}} + \lambda \dfrac{\partial j}{\partial \mathbf{x}} = \mathbf{0} \\ \dfrac{\partial \mathbf{u}}{\partial \mathbf{y}} + \lambda \dfrac{\partial j}{\partial \mathbf{y}} = \mathbf{0} \\ j(\mathbf{x,y}) = \mathbf{0} \end{cases} \Leftrightarrow \begin{cases} \mathbf{\dfrac{1}{2} \dfrac{y+1}{\sqrt{(x+2)(y+1)}} + \lambda 4 = 0} \\ \mathbf{\dfrac{1}{2} \dfrac{x+2}{\sqrt{(x+2)(y+1)}} + \lambda 6 = 0} \\ \mathbf{4x + 6y - 200 = 0} \end{cases}$$

giai he ñöôc: $x_0 = \mathbf{\dfrac{99}{4}}$; $y_0 = \mathbf{\dfrac{101}{6}}$; $\lambda = \mathbf{\dfrac{-1}{4\sqrt{6}}}$. ==> u= $\mathbf{\dfrac{107}{\sqrt{24}}}$

Xet ham Lagrange: L(x,y) = u(x,y) + $\lambda\varphi$(x,y) $= \mathbf{\sqrt{(x+2)(y+1)} - \dfrac{1}{4\sqrt{6}}(4x+6y-200)}$, tính toan: $A=\dfrac{\partial^2 L}{\partial x^2}=-\dfrac{(y+1)^2}{4\left(\sqrt{(x+2)(y+1)}\right)^3}=-\dfrac{1}{4}\left(\dfrac{\sqrt{y+1}}{\sqrt{(x+2)^3}}\right)$,

$B=\dfrac{\partial^2 L}{\partial x\partial y}=\dfrac{1}{4}\dfrac{1}{\sqrt{(x+2)(y+1)}}$; $C=\dfrac{\partial^2 L}{\partial y^2}=-\dfrac{1}{4}\dfrac{\sqrt{x+2}}{\sqrt{(y+1)^3}}$,

$$A=-\frac{1}{428}\sqrt{\frac{32}{3}}=\frac{-\sqrt{6}}{312};\quad C=-\frac{\sqrt{54}}{428}=\frac{-3\sqrt{6}}{428};\quad B=\frac{\sqrt{24}}{428}=\frac{\sqrt{6}}{214}$$

$$d^2L=-\frac{1}{428}\sqrt{\frac{32}{3}}dx^2+2\frac{1}{428}\sqrt{24}dxdy-\frac{1}{428}\sqrt{54}dy^2$$

dx, dy thoa $d\varphi = 0 \Leftrightarrow 4dx + 6dy = 0 \Leftrightarrow dy = -2/3\,dx$

$\Rightarrow$ $d^2L < 0$ $\Rightarrow$ u ñat cöc ñai

Cach khac1: Viet lai döôi dang vi phan:

$$d^2L=-\frac{3\sqrt{6}}{428}\left(dy-\frac{\sqrt{6}}{18}dx\right)^2-\frac{23\sqrt{6}}{7704}dx^2<0\Rightarrow$$ u ñat cöc ñai

Cach khac2: $\dfrac{4(x+2)+6(y+1)}{2}\geq\sqrt{4(x+2).6(y+1)}=\sqrt{24u}$

$$107=\frac{(4x+6y-200)+214}{2}=\frac{4(x+2)+6(y+1)}{2}\geq\sqrt{24u}\Leftrightarrow\frac{107}{\sqrt{24}}\geq u$$

va chæ bang khi $4(x+2)=6(y+1)\Rightarrow x=\dfrac{99}{4},\ y=\dfrac{101}{6}$

Cach khac3: Co the giai bang hình hoc, sinh vien töï giai.

Cach khac4: Dung ñònh thöc xet dau: n=2, m=1, chæ xet mot ñònh thöc:

$$H_2=\begin{vmatrix}0 & j'_x & j'_y\\ j'_x & a_{11} & a_{12}\\ j'_y & a_{21} & a_{22}\end{vmatrix}=\begin{vmatrix}0 & 4 & 6\\ 4 & \frac{-\sqrt{6}}{321} & \frac{\sqrt{6}}{214}\\ 6 & \frac{\sqrt{6}}{214} & \frac{-3\sqrt{6}}{428}\end{vmatrix}=\frac{48\sqrt{6}}{107}$$ >0 ==> CÑ

4.4 Bai toan tìm ñau vao sao cho chi phí san xuat be nhat.

Gia söû ham san xuat Q = Q(x,y) co ñao ham rieng ñen cap 2 lien tuc va $\dfrac{\partial Q}{\partial x}>0;\dfrac{\partial Q}{\partial y}>0$. Ñôn gia cua ñau vao co ñònh ky hieu la: p_1,p_2. Bai toan ñat ra la: xac ñònh <u>san löôïng ñau vao</u> (x,y) ñe san xuat ra Q san pham vôùi tong chi phí be nhat.

Mo hình cua bai toan la:

Tìm (x_0,y_0) ñe $C = p_1x + p_2y$ nho nhat vôùi ñieu kien

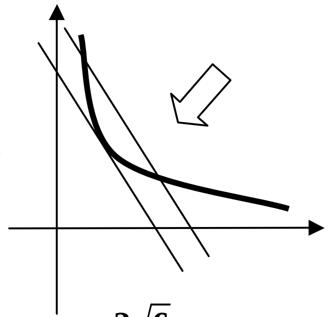
Q(x,y) = Q. Ta phai tìm gia tri nho nhat cua ham

$C = C(x,y) = p_1x + p_2y$ thoa ñieu kien: $\varphi(x,y) = Q(x,y) - Q = 0$.

Ñay la bai toan cöïc trò co ñieu kien ña biet cach giai.

Ví du1: Cho $Q(x,y) = 10\sqrt{xy}$; $P_1 = 10$; $P_2 = 15$; $Q=20$

$$\begin{cases}\dfrac{\partial \mathbf{C}}{\partial \mathbf{x}}+\lambda\dfrac{\partial j}{\partial \mathbf{x}}=\mathbf{0}\\ \dfrac{\partial \mathbf{C}}{\partial \mathbf{y}}+\lambda\dfrac{\partial j}{\partial \mathbf{y}}=\mathbf{0}\\ j\,(\mathbf{x,y})=\mathbf{10}\sqrt{\mathbf{xy}}-\mathbf{20}=\mathbf{0}\end{cases}\Leftrightarrow\begin{cases}\mathbf{10}+\lambda\mathbf{5}\sqrt{\dfrac{\mathbf{y}}{\mathbf{x}}}=\mathbf{0}\\ \mathbf{15}+\lambda\mathbf{5}\sqrt{\dfrac{\mathbf{x}}{\mathbf{y}}}=\mathbf{0},\\ \sqrt{\mathbf{xy}}=2\end{cases}$$



Giai he tìm ñöôc (chæ lay nghiem döông): $\mathbf{x_0}=\sqrt{6},\mathbf{y_0}=\dfrac{2\sqrt{6}}{3},\lambda=-\sqrt{6}$,

Xet ham Lagrange:

L(x,y) = C(x,y) + λφ(x,y)= 10x + 15y - $\sqrt{6}\left(\mathbf{10}\sqrt{\mathbf{xy}}-\mathbf{20}\right)$.

$$\frac{\partial^2\mathbf{L}}{\partial\mathbf{x}^2}=\frac{5\sqrt{6}}{2}.\frac{1}{\mathbf{x}}\sqrt{\frac{\mathbf{y}}{\mathbf{x}}};\quad \frac{\partial^2\mathbf{L}}{\partial\mathbf{x}\partial\mathbf{y}}=\frac{-5\sqrt{6}}{2}.\frac{1}{\sqrt{\mathbf{xy}}};\quad \frac{\partial^2\mathbf{L}}{\partial\mathbf{y}^2}=\frac{5\sqrt{6}}{2}.\frac{1}{\mathbf{y}}\sqrt{\frac{\mathbf{x}}{\mathbf{y}}}$$

$$\Rightarrow \mathbf{A}=\frac{5\sqrt{6}}{2}.\frac{1}{\sqrt{6}}\sqrt{\frac{2\sqrt{6}/3}{\sqrt{6}}}=\frac{5}{2}\sqrt{\frac{2}{3}}=\frac{5\sqrt{6}}{6};\quad \mathbf{B}=\frac{-5\sqrt{6}}{2}.\frac{1}{2}=\frac{-5\sqrt{6}}{4};$$

$$\mathbf{C}=\frac{5\sqrt{6}}{2}\frac{3}{2\sqrt{6}}\sqrt{\frac{3}{2}}=\frac{15}{4}\sqrt{\frac{3}{2}}=\frac{15\sqrt{6}}{8}$$

vay $\Delta=\mathbf{AC\text{-}B}^2=\dfrac{75}{8}-\dfrac{150}{16}=\mathbf{0}\Rightarrow$ L(x,y) ñat cöc tieu ?? (pha san)

Cach khac: Viet lai döôi dang vi phan:

$d^2L=\dfrac{5}{2}\sqrt{\dfrac{2}{3}}dx^2+\dfrac{15}{4}\sqrt{\dfrac{3}{2}}dy^2-\dfrac{5}{2}\sqrt{6}dxdy$, sau mot phep tính toan, ta co:

$d^2L=\dfrac{15}{8}\sqrt{6}\left(dy-\dfrac{2}{3}dx\right)^2>0\Rightarrow$ L(x,y) ñat cöc tieu ?. That ra ta co the ñöa ve ham mot bien ñe khao sat, ñe thay thöc sö L(x,y) ñat cöc tieu. Co the thay ro ñieu nay qua y nghóa hình hoc cua bai toan nhö hình ve. ta con co the thay ñöôc ket qua nay qua BÑT Cauchy:

$$\frac{10x+15y}{2}\ge\sqrt{\mathbf{10x}\times\mathbf{15y}}=\sqrt{150}\sqrt{xy}=2\sqrt{150}\Rightarrow 10x+15y\ge4\sqrt{150},$$

BÑT chæ xay ra dau bang khi $10x=15y\Leftrightarrow 2x=3y\Rightarrow \mathbf{x}=\sqrt{6},\mathbf{y}=\dfrac{2\sqrt{6}}{3}$

Nhan xet: Ñay la bai toan co dang vi phan bac 2 toan phöông la suy bien nghia la $|H|=0\Leftrightarrow\Delta=\mathbf{AC\text{-}B}^2=\dfrac{75}{8}-\dfrac{150}{16}=\mathbf{0}$, nhöng ñiem döng van la ñiem cöc trò nhö ña nhan xet ô tren.

Cach khac: Dung ñònh thöc xet dau: n=2, m=1, chæ xet mot ñònh thöc:

$$H_2=\left|\begin{array}{c|cc}0&\dfrac{\partial g}{\partial x_1}&\dfrac{\partial g}{\partial x_2}\\ \hline \dfrac{\partial g}{\partial x_1}&a_{11}&a_{12}\\ \dfrac{\partial g}{\partial x_2}&a_{21}&a_{22}\end{array}\right|=\left|\begin{array}{c|cc}0&\dfrac{10}{\sqrt{6}}&\dfrac{15}{\sqrt{6}}\\ \hline \dfrac{10}{\sqrt{6}}&\dfrac{5\sqrt{6}}{6}&\dfrac{-5\sqrt{6}}{4}\\ \dfrac{15}{\sqrt{6}}&\dfrac{-5\sqrt{6}}{4}&\dfrac{15\sqrt{6}}{8}\end{array}\right|=-306.18<0 \Longrightarrow \text{CT.}$$

Ví duï2: *Ñeà thi cao hoïc kinh teá 5/2007.*

Duøng phöông phaùp Lagrange tìm löôïng lao ñoäng L (Labor) vaø voán K (Capital), ñeå cöïc tieåu hoùa haøm chi phí C (Cost), $C = L + 0.01K$, K>0, L>0, vôùi raøng buoäc veà haøm saûn xuaát Cobb-Douglas: $Q = \sqrt{LK} = 20$.

Giaûi:

$f(L,K,l) = L + 0.01K + l\left(\sqrt{LK} - 20\right)$, $g(L,K) = \sqrt{LK} - 20 = 0$

$f'_L = 1 + l\dfrac{K}{2\sqrt{LK}} = 0$, $f'_K = 0.01 + l\dfrac{L}{2\sqrt{LK}} = 0$, $f'_l = \sqrt{LK} - 20 = 0$

$\Rightarrow 100 = \dfrac{1}{0.01} = \dfrac{K}{L} \Rightarrow K = 100L$,

theá vaøo $\sqrt{LK} = 20 \Rightarrow \sqrt{L \times 100L} = 20 \Leftrightarrow L = 2$, K=200, $l = -0.2$

vì K>0, L>0, neân ta khoâng laáy nghieäm aâm.

$f''_{LL} = -\dfrac{1}{4} l K^2 (LK)^{-3/2}$, $f''_{KK} = -\dfrac{1}{4} l L^2 (LK)^{-3/2}$

$f''_{LK} = -\dfrac{1}{4} l KL(LK)^{-3/2} + \dfrac{1}{2} l \sqrt{KL}$, $f''_{ll} = 0$,

$f''_{Ll} = \dfrac{K}{2\sqrt{LK}}$, $f''_{Kl} = \dfrac{L}{2\sqrt{LK}}$, $g'_L = K^{1/2}\dfrac{1}{2}L^{-1/2} = \dfrac{1}{2}\sqrt{\dfrac{K}{L}}$, $g'_K = \dfrac{1}{2}\sqrt{\dfrac{L}{K}}$

theá 3 giaù trò L=2, K=200, $l = -0.2$

$f''_{LL} =$ 0.25, $f''_{KK} =$ 0.000025, $f''_{LK} =$ - 0.0025. Theá vaøo ma traän Hessian:

$$H_2 = \left|\begin{array}{c|cc} 0 & \dfrac{\partial g}{\partial L} & \dfrac{\partial g}{\partial K} \\ \hline \dfrac{\partial g}{\partial L} & a_{11} & a_{12} \\ \dfrac{\partial g}{\partial K} & a_{21} & a_{22} \end{array}\right| = \left|\begin{array}{c|cc} 0 & 5 & 0.05 \\ \hline 5 & 0.25 & -0.0025 \\ 0.05 & -0.0025 & 0.000025 \end{array}\right| = -0.0025 < 0 \text{ ==> CT.}$$

Ta coù theå duøng vi phaân baäc 2:

$$d^2 f = f''_{LL} dL^2 + 2 f''_{LK} dLdK + f''_{KK} dK^2 = 0.25(dL - 0.01dK)^2 > 0 \text{ ==> CT.}$$

4.5 Coâng ty coù nhieàu thò tröôøng taùch bieät.

Phaàn naøy toång quaùt hôn phaàn 4.1 vaø 4.2 ôû treân.

Moät coâng ty saûn xuaát ñoäc quyeàn moät loaïi saûn phaåm, tieâu thuï treân 3 thò tröôøng rieâng bieät. caùc haøm caàu (Demand) veà saûn löôïng (Quantity) theo ñôn giaù (Price) töông öùng treân 3 thò tröôøng

$Q_{D_1} = Q_1(P) = D_1(P_1,P_2,P_3)$, $Q_{D_2} = Q_2(P) = D_2(P_1,P_2,P_3)$, $Q_{D_3} = Q_3(P) = D_3(P_1,P_2,P_3)$,

Thoâng thöôøng vôùi $i = \overline{1,2,3}$, $P_i \uparrow \Rightarrow Q_{D_i}(P) \downarrow$ hay ngöôïc laïi.

Va ham chi phí C=C(Q), vôi $Q=Q_1+Q_2+Q_3$. Thong thöông C=C(Q) la ham bac 2 cua Q, hay la mot <u>dang toan phöông döông</u> tong quat theo 3 bien san löông: Q_1,Q_2,Q_3 nhö

$C(Q)=C(Q_1,Q_2,..,Q_n)=Q^T AQ+BQ+const$.

•Tìm so löông san pham (Q_1,Q_2,Q_3), cung cap cho töng thò tröông ñe <u>lôi nhuan cao nhat</u> doanh thu (Revenue):

$R_1(Q_1)=P_1(Q_1)\times Q_1$, $R_2(Q_2)=P_2(Q_2)\times Q_2$, $R_3(Q_3)=P_3(Q_3)\times Q_3$.

Lôi nhuan (Profit): $\boxed{p=R-C}=R_1(Q_1)+R_2(Q_2)+R_3(Q_3)-C(Q)$.

trong ño doanh thu: $R=\sum_{i=1}^{3}R_i(Q_i)=R_1(Q_1)+R_2(Q_2)+R_3(Q_3)$.

• Vì P_i luon trai dau vôi Q_i, ve hình thöc $R_i=P_iQ_i=-Q_i^2$ va $C(Q)=Q^T AQ$>0 nen $p=R-C=-Q_i^2-Q^T AQ$ co dang eliptic $-xQ_1^2-yQ_2^2-zQ_3^2+A\to \max=A$ vôi $x,y,z,A>0$: luon **à** max.

Ta ñònh nghia them mot so khai niem trong toan kinh te nhö sau:

• Bien te chi phí MC (Marginal Cost): $MC=C_Q^/$ (ñao ham rieng cua chi phí C theo bien san löông Q).

• Bien te doanh thu MR (Marginal Revenue): $MR=R_Q^/$ (ñao ham rieng cua doanh thu theo bien san löông Q).

• Ño co dan: $e_{Q(P)}=\dfrac{\frac{\Delta Q}{\Delta P}}{\frac{Q}{P}}=\dfrac{\Delta Q}{\Delta P}\dfrac{P}{Q}\xrightarrow{\Delta P\to 0}Q_P^/\dfrac{P}{Q}$. trong ño ham $Q=Q(P)$. Vay $e_{Q(P)}=Q_P^/\dfrac{P}{Q}$.

Nen khi tai ñôn gia P, neu ta thay ñoi 1% ñôn gia, thì san löông cau Q thay ñoi theo chieu höông tang hay giam nhô vao ño co dan $e_{Q(P)}\%$ döông hay am.

Ví du: tai mot ñôn gia P nao ño va sau khi tính ta co: $e_{Q(P)}=-5$, nghia la neu ta tang them 1% ñôn gia, thì san löông cau Q se giam 5%. Ngöôc lai neu ta giam them 1% ñôn gia, thì san löông cau Q se tang 5%.

<u>Ví du0:</u> Mot cong ty ban mot loai san pham vôi ham cau: $Q_D=D\ (P\ )=200-50P$, tong chi phí la $C(Q)=Q+10$, trong ño Q la san löông, P la ñôn gia.

a/ Hay xac ñònh ham lôi nhuan va tìm Q ñe cong ty ñat lôi nhuan toi ña.
b/ Tính chi phí bien te MC, va doanh thu bien te MR, so sanh giöa chung.
c/ Tính ño co dan cua Q theo P tai P=3.6. Neu tai ñôn gia P=3.6 nay, cong ty muon tang ñôn gia them 1%, thì san löông cau tang hay giam bao nhieu phan tram.

<u>Giai:</u>

a/ Ta co: $P=4-0.02Q$, ==> $R=QP=-0.02Q^2+4Q$,

$p=R-C=-0.02Q^2+3Q-10$, $\Rightarrow p_Q^/=-0.04Q+3=0\Rightarrow Q=75$,

$p_Q^{//}=-0.04<0$, vay lôi nhuan p ñat cöc ñai tai Q=75 san löông.

b/ Tö $C(Q)=Q+10$ ==> $MC=C_Q^/$=1. Tö $R=QP=-0.02Q^2+4Q$

==> $MR = R_Q^{/} = -0.04Q + 4$, khi Q=75 thì MR=1, luùc naøy MC=MR.

c/ Taïi P=3.6, $e_{Q(P)} = Q_P^{/}\frac{P}{Q} = -50\frac{P}{200-50P} = -50\frac{3.6}{200-50\times 3.6} = -9$

neân khi taïi ñôn giaù P=3.6, neáu ta taêng ñôn giaù theâm 1% , thì saûn löôïng caàu Q seõ giaûm 9%.

Ví duï1: Moät xí nghieäp saûn xuaát ñoäc quyeàn 2 maët haøng vôùi caùc chæ tieâu sau:

$$\begin{cases} Q_{D1} = Q_1 = \boxed{-2P_1} + 1.P_2 + 800 \\ Q_{D2} = Q_2 = 1.P_1 \quad \boxed{-3P_2} + 900 \end{cases}$$, vôùi $Q = Q_1 + Q_2$, chi phí:

$C(Q) = 320Q_1 + 250Q_2 + 300$, tìm saûn löôïng ñeå lôïi nhuaän laø toái ña.

Giaûi:

Töø heä treân, giaûi ra: $P_1 = 660 - \frac{3Q_1}{5} - \frac{Q_2}{5}$, $P_2 = 520 - \frac{Q_1}{5} - \frac{2Q_2}{5}$

vôùi $p = P_1Q_1 + P_2Q_2 - (320Q_1 + 250Q_2 + 300) =$

$$= \left[660 - \frac{3Q_1}{5} - \frac{Q_2}{5}\right]Q_1 + \left[520 - \frac{Q_1}{5} - \frac{2Q_2}{5}\right]Q_2 - [320Q_1 + 250Q_2 + 300]$$

$$= 340Q_1 - \frac{3}{5}Q_1^2 - \frac{2}{5}Q_1Q_2 + 270Q_2 - \frac{2}{5}Q_2^2 - 300$$

(daïng eliptic $-xQ_1^2 - yQ_2^2 + A \to \max = A$ vôùi $x, y, A > 0$: luoân à max)

$$\begin{cases} p_{Q_1}^{/} = \frac{-6}{5}Q_1 - \frac{2}{5}Q_2 + 340 = 0 \Leftrightarrow \frac{6}{5}Q_1 + \frac{2}{5}Q_2 = 340 \\ p_{Q_2}^{/} = \frac{-2}{5}Q_1 - \frac{4}{5}Q_2 + 270 = 0 \Leftrightarrow \frac{2}{5}Q_1 + \frac{4}{5}Q_2 = 270 \end{cases}$$ <==> $Q_1 = 205$,

$Q_2 = 235$, cöïc trò vöøng: $\begin{vmatrix} A & B \\ B & C \end{vmatrix} = \frac{24 - 4}{25} > 0$ vaø $A = \frac{-6}{5} < 0$, neân CÑ

Ví duï2: Giaû thuyeát nhö treân vôùi: $Q_{D_1} = D_1(P_1) = 80 - \frac{P_1}{2}$,

$Q_{D_2} = D_2(P_2) = 80 - \frac{P_2}{3}$, $Q_{D_3} = D_3(P_3) = 80 - \frac{P_3}{4}$, $Q = Q_1 + Q_2 + Q_3$

Chi phí: $C(Q) = Q^2 + 10Q + 30$, chi phí naøy chính laø moät bieåu dieãn quadric toång quaùt trong R^3 nhö ñaõ phaân tích baøn ñaàu (vöøa coù daïng toaøn phöông, vöøa coù daïng baäc nhaát tuyeán tính, vöøa coù haèng soá ban ñaàu).

Giaûi:

Nhaän thaáy: vôùi $i = \overline{1,2,3}$, $P_i \uparrow \Rightarrow Q_{D_i} \downarrow$ hay ngöôïc laïi.

Ta coù: $P_1 = 160 - 2Q_1$, $P_2 = 240 - 3Q_2$, $P_3 = 320 - 4Q_3$,

==> $R_1(Q_1) = P_1(Q_1)Q_1 = (160 - 2Q_1)Q_1$

==> $R_2(Q_2) = P_2(Q_2)Q_2 = (240 - 3Q_2)Q_2$

==> $R_3(Q_3) = P_3(Q_3)Q_3 = (320 - 4Q_3)Q_3$

vôùi $Q = Q_1 + Q_2 + Q_3$

$p = R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q) = R_1 + R_2 + R_3 - Q^2 - 10Q - 30$

Chu y: $\left(Q^2+10Q+30\right)'_{Q_i} = 2QQ'_{Q_i} + 10Q'_{Q_i} = 2Q\times 1 + 10\times 1 = 2Q+10$

$$\overrightarrow{\nabla p} = \vec{0} \Leftrightarrow \begin{cases} p'_{Q_1} = 0 \\ p'_{Q_2} = 0 \\ p'_{Q_3} = 0 \end{cases} \Leftrightarrow \begin{cases} 160-4Q_1 = 2Q+10 = 2(Q_1+Q_2+Q_3)+10 \\ 240-6Q_2 = 2Q+10 = 2(Q_1+Q_2+Q_3)+10 \\ 320-8Q_3 = 2Q+10 = 2(Q_1+Q_2+Q_3)+10 \end{cases}$$

$$\Leftrightarrow \begin{cases} 3Q_1+Q_2+Q_3 = 75 \\ Q_1+4Q_2+Q_3 = 115 \\ Q_1+Q_2+5Q_3 = 155 \end{cases} \Leftrightarrow \begin{cases} Q_1 = 10 \\ Q_2 = 20 \\ Q_3 = 25 \end{cases}$$. Xet ñao ham bac 2:

$p''_{Q_1Q_1} = -6$, $p''_{Q_2Q_2} = -8$, $p''_{Q_3Q_3} = -10$, $p''_{Q_1Q_2} = p''_{Q_1Q_3} = p''_{Q_2Q_3} = -2$

$H = \begin{bmatrix} -6 & -2 & -2 \\ -2 & -8 & -2 \\ -2 & -2 & -10 \end{bmatrix}$, co $H_1 = -6$, $H_2 = \begin{vmatrix} -6 & -2 \\ -2 & -8 \end{vmatrix} = 44 > 0$,

$H_3 = \begin{vmatrix} -6 & -2 & -2 \\ -2 & -8 & -2 \\ -2 & -2 & -10 \end{vmatrix} = -2^3 \begin{vmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{vmatrix} = -8\times 50 < 0$, nen p CÑ

Vay can so lööng Q_1=10, töông öng gia la: $P_1 = 160-2Q_1$=140

Vay can so lööng Q_2=20, töông öng gia la: $P_2 = 240-3Q_2$=180

Vay can so lööng Q_3=25, töông öng gia la: $P_3 = 320-4Q_3$=220

Cach khac: Viet lai döôi dang vi phan:

$$d^2p = -6dQ_1^2 - 8dQ_2^2 - 10dQ_3^2 - 4dQ_1dQ_2 - 4dQ_1dQ_3 - 4dQ_2dQ_3 =$$

$$= -10\left(dQ_3 + \frac{dQ_2+dQ_1}{5}\right)^2 - \frac{38}{5}\left(dQ_2 + \frac{4dQ_1}{19}\right)^2 - \frac{100}{19}dQ_1^2 < 0$$

Nen p CÑ tai (Q_1=10, Q_2=20, Q_3=25).

_Ví du3:_ Cung vôi gia thuyet nhö tren vôi:

$Q_{D1} = Q_1 = \boxed{-3P_1} + P_2 + 0P_3 + 130$,

$Q_{D2} = Q_2 = 2P_1\boxed{-7P_2} + P_3 + 220$

$Q_{D3} = Q_3 = 0P_1 + P_2\boxed{-5P_3} + 215$. Chi phí: $C(Q) = Q^2 - 10Q + 30$,

Giai:

Nhan thay: vôi $i = \overline{1,2,3}$, $P_i \uparrow \Rightarrow Q_{D_i} \downarrow$ hay ngöôc lai.

$p = R - C = P_1Q_1 + P_2Q_2 + P_3Q_3 - \left(Q^2 - 10Q + 30\right)$

Ta co the giai theo P_1, P_2, P_3, roi chuyen ve Q_1, Q_2, Q_3

$Q = Q_1 + Q_2 + Q_3 = -P_1 - 5P_2 - 4P_3 + 565$

$Q = Q_1 + Q_2 + Q_3 \Rightarrow Q'_{P_1} = Q'_{1,P_1} + Q'_{2,P_1} + Q'_{3,P_1} = -3+2+0 = -1$,

$$\bullet p'_{P_1} = \left(P_1 Q'_{1,P_1} + Q_1\right) + P_2 Q'_{2,P_1} + P_3 Q'_{3,P_1} - \left(2QQ'_{P_1} - 10Q'_{P_1}\right) =$$

$$(-6P_1 + P_2 + 130) + 2P_2 + 0P_3 - 2(-P_1 - 5P_2 - 4P_3 + 565)(-1) + 10(-1)$$

$$= (-6P_1 + P_2 + 130) + 2P_2 + 2(-P_1 - 5P_2 - 4P_3 + 565) - 10$$

$p'_{P_1} = (-8P_1 - 7P_2 - 8P_3 + 1250)$=0, töông töi:

$$\bullet p'_{P_2} = P_1 Q'_{1,P_2} + \left(P_2 Q'_{2,P_2} + Q_2\right) + P_3 Q'_{3,P_2} - \left(2QQ'_{P_2} - 10Q'_{P_2}\right) =$$

$$Q'_{P_2} = Q'_{1,P_2} + Q'_{2,P_2} + Q'_{3,P_2} = 1 - 7 + 1 = -5,$$

$p'_{P_2} = (-7P_1 - 64P_2 - 38P_3 + 5820)$=0, töông töi:

$$\bullet p'_{P_3} = P_1 Q'_{1,P_3} + P_2 Q'_{2,P_3} + \left(P_3 Q'_{3,P_3} + Q_3\right) - \left(2QQ'_{P_3} - 10Q'_{P_3}\right) =$$

$$Q'_{P_3} = Q'_{1,P_3} + Q'_{2,P_3} + Q'_{3,P_3} = 0 + 1 - 5 = -4,$$

$p'_{P_3} = (-8P_1 - 38P_2 - 42P_3 + 4695)$=0, baim may ta coi:

giai ra: $P_1 = \dfrac{229115}{4027},\ P_2 = \dfrac{215590}{4027},\ P_3 = \dfrac{422925}{8054}$, thei vaio ta coi:

$Q_1 = \dfrac{51755}{4027},\ Q_2 = \dfrac{93005}{8054},\ Q_3 = \dfrac{48165}{8054}$. Xet ñaio ham bac 2:

$p''_{P_1P_1} = -8$, $p''_{P_2P_2} = -64$, $p''_{P_3P_3} = -42$,

$p''_{P_1P_2} = -7,\ p''_{P_1P_3} = -8,\ p''_{P_2P_3} = -38$, vai ma train Hess:

$$H = \begin{bmatrix} -8 & -7 & -8 \\ -7 & -64 & -38 \\ -8 & -38 & -42 \end{bmatrix}, \text{ coi } H_1 = -8 < 0,\ H_2 = \begin{vmatrix} -8 & -7 \\ -7 & -64 \end{vmatrix} > 0,$$

$$H_3 = \begin{vmatrix} -8 & -7 & -8 \\ -7 & -64 & -38 \\ -8 & -38 & -42 \end{vmatrix} = -5566 < 0$$, nein p CÑ

Chöông 2: TÍCH PHAIN BOII 2 (KEIP)

BAII 1: BAII TOAIN MÔI ÑAIU

1.1 Bai toain môi ñaiu :

Ta nhôi laii raing, tích phain xaic ñònh hay tích phain ñôn phait xuait töi bai toain tính diein tích vai tích phain kep laii xuait phait töi bai toain tính thei tích nhö sau :

Cain tính thei tích cuia hình coi ñaiy phaing lai mien D $\subset$ Oxy, ñöôøng sinh song song vôi truic oz, mait trein lai mait cong xaic ñònh bôi ham

z = z(x,y), vôi M(x,y) $\in$ D. Ta chia mien D thanh n phain, dieïn tích moii phain lai $\Delta(S_i)$, coit thei tích coi dieïn tích ñaiy

$\Delta(S_i) \ni M_i(x_i,y_i)$, vay cai thei tích lai :

$$V_n = \sum_{i=1}^{n} (\Delta S_i) f(M_i) = \sum_{i=1}^{n} (\Delta S_i) f(x_i, y_i)$$

$V = \lim\limits_{\max d_i \to 0} V_n$, vôi d_i = dia(ΔS_i) = ban kính cua ΔS_i

chu y : max $d_i \to 0 \;\Rightarrow\; n \to \infty$

1.2 Ñònh nghóa :

Giôi han tren neu ton tai höu han (khong phu thuoc vao cach chia mien D) ñöôc goi la tích phan kep cua ham z = f(x,y) tren mien D, khi ño ta noi f kha tích tren mien D va ky hieu noi la

$$\iint_D f(x,y)dxdy$$

<u>Nhan xet</u> *:*

1) Theo ñònh nghóa neu f(x,y) = 1 $\forall$(x,y) $\in$ D thì $\iint_D 1dxdy = S(D)$ (dien tích mien D).

2) f(x,y) > 0, lien tuc $\forall$(x,y) $\in$ D thì $\iint_D f(x,y)dxdy = V(D)$ la the tích hình tru co cac ñöông sinh song song vôi Oz, hai ñay giôi han bôi z= 0, z= f(x,y).

1.3 Tính chat cua tích phan kep :

Cac tính chat cua tích phan kep cung giong nhö tính chat cua tích phan xac ñònh.

<u>Tính chat 1</u> : f lien tuc trong D thì f kha tích tren D

<u>Tính chat 2</u> : co tính tuyen tính

$$\iint_D (f+g)dxdy = \iint_D fdxdy + \iint_D gdxdy$$

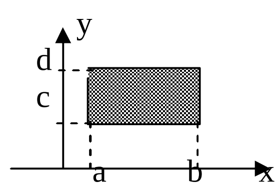
$$\iint_D Kfdxdy = K\iint_D fdxdy \quad , K \in R$$

<u>Tính chat 3</u> : neu D = $D_1 \cup D_2$ va $D_1 \cap D_2 = \varnothing$ thì

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f$$

BAI 2: PHÖÔNG PHAP TÍNH TÍCH PHAN KEP

2.1 Trong he truc Descartes.



- *Mien D la hình chö nhat :*

D = {(x,y) : a ≤ x ≤ b, c ≤ y ≤ d } =[a,b]× [c,d]

$$\iint_D f(x,y)dxdy = \int_c^d\left(\int_a^b f(x,y)dx\right)dy =$$

$$= \int_a^b\left(\int_c^d f(x,y)dy\right)dx$$

Vay vôi mien la hình chö nhat ta co the hoan vò can .

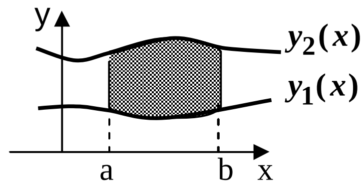
<u>Ví du :</u> D = [0,1]×[1,2]

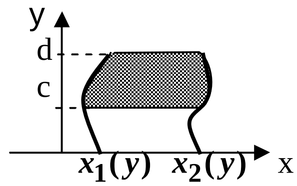
$$I = \iint_D (x^2+y^2)dxdy = \int_0^1\left(\int_1^2 (x^2+y^2)dy\right)dx$$

$$= \int_0^1 (x^2 y + \frac{y^3}{3})\Big|_1^2 dx = \int_0^1 (x^2 + \frac{7}{3})dx = \frac{8}{3}, \quad \text{hoaëc}$$

$$I = \int_1^2 (\int_0^1 (x^2 + y^2)dx)dy = \int_1^2 (\frac{x^3}{3} + xy^2)\Big|_0^1 dy = \int_1^2 (\frac{1}{3} + y^2)dy = \frac{y}{3} + \frac{y^3}{3}\Big|_1^2 = \frac{8}{3}$$

q *Mieàn D laø mieàn baát kyø :*





- D = {(x,y) : a ≤ x ≤ b, $y_1(x) \le y \le y_2(x)$ } =[a,b]× [$y_1(x)$, $y_2(x)$]

$$\iint_D f(x,y)dxdy = \int_a^b dx \int_{y=y_1(x)}^{y=y_2(x)} f(x,y)dy$$

- D = {(x,y) : c ≤ y ≤ d, $x_1(y) \le x \le x_2(y)$ } =[$x_1(y)$, $x_2(y)$]× [c,d]

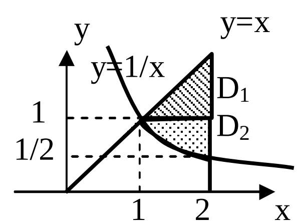
$$\iint_D f(x,y)dxdy = \int_{y=c}^{y=d} dy \int_{x=x_1(y)}^{x=x_2(y)} f(x,y)dx$$

- Neáu D laø mieàn phöùc taïp thì phaûi phaân D ra thaønh nhöõng mieàn ñôn giaûn nhö treân

<u>*Ví duï 1 :*</u> tính $I = \iint_D \frac{x^2}{y^2}dxdy$

$D = D_1 \cup D_2$

Vôùi D giôùi haïn bôûi 3 ñöôøng sau : $x = 2$, $y = \frac{1}{x}$, $y = x$



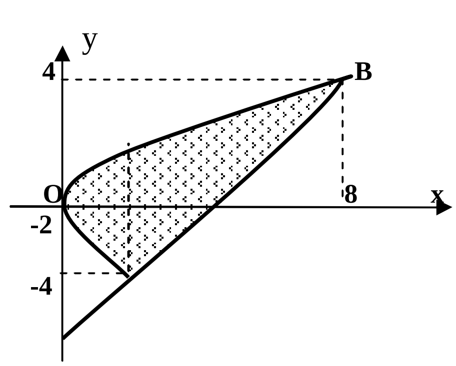
<u>caùch 1</u> : $I = \int_1^2 dx \int_{y=\frac{1}{x}}^{y=x} \frac{x^2}{y^2}dy = \frac{9}{4}$

<u>caùch 2</u> : $I = \iint_{D_2} + \iint_{D_2}$ vì $D = D_1 \cup D_2$ vaø $D_1 \cap D_2 = \varnothing$

$$\iint_{D1} = \int_{y=1}^{y=2} dy \int_{x=y}^{x=2} \frac{x^2}{y^2}dx = \frac{-8}{3y} - \frac{y^2}{6}\Big|_1^2 = \frac{5}{6}$$

$$\iint_{D2} = \int_{\frac{1}{2}}^{1} dy \int_{x=\frac{1}{y}}^{x=2} \frac{x^2}{y^2}dx = \frac{-8}{3y} + \frac{1}{12}y^{-4}\Big|_{\frac{1}{2}}^{1} = \frac{8}{3} - \frac{15}{12}$$

$$\Rightarrow \quad I = \frac{5}{6} + \frac{8}{3} - \frac{15}{12} = \frac{9}{4}$$



<u>*Ví duï 2 :*</u> Tính $I = \iint_D xydxdy$,

D giôi han bôi

2 ñöông y = x – 4 va $y^2 = 2x$,

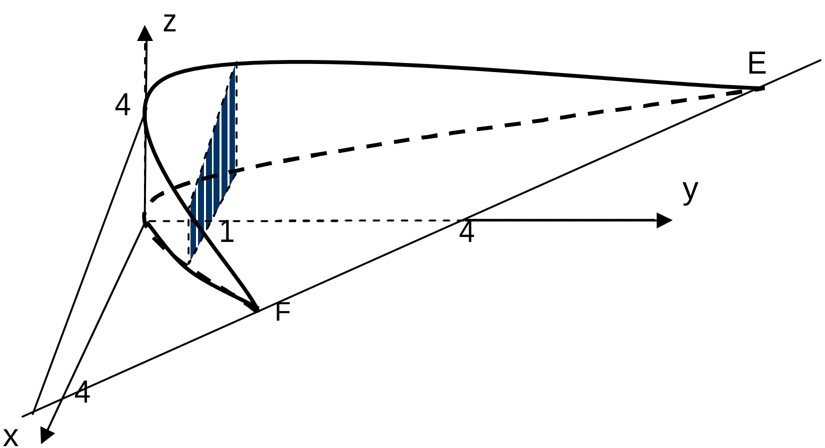
Tính giao ñiem ta co :

giao ñiem : A(2,-2), B(8,4)

$$I = \int_{-2}^{4} dy \int_{x=\frac{y^2}{2}}^{x=y+4} xydx = \int_{-2}^{4} \left(y\frac{x^2}{2} \Bigg|_{\frac{y^2}{2}}^{y+4} \right) dy = \frac{1}{2}\left(\frac{y^4}{4} + \frac{8}{3}y^3 + 8y^2 - \frac{y^6}{24}\right)\Bigg|_{-2}^{4} = 90$$

<u>*Cach 2 :*</u> ñoi can danh cho ban ñoc coi nhö bai tap.

<u>*Ví du 3 :*</u> Tính the tích khoi, ñöôc giôi han bôi cac mat z= 4-x-y va $z \geq 0$, $y=x^2$, $y \leq 1$.

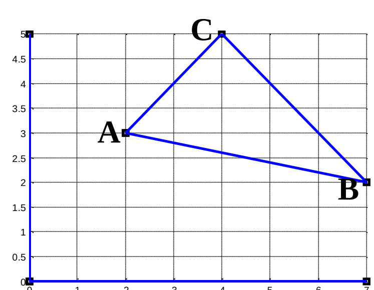


V2= $\iint_D (4-x-y)dxdy = \int_{y=0}^{y=1} dy \int_{x=-\sqrt{y}}^{x=+\sqrt{y}} (4-x-y)dx = \int_{y=0}^{y=1} (8\sqrt{y} - 2y^{\frac{3}{2}})dy = \frac{68}{15}$

Cach 2: Ñoi can khac: $V_2 = \int_{x=-1}^{x=1} dx \int_{y=x^2}^{y=2} (4-x-y)dy = \frac{68}{15}$

<u>*Ví du 7 :*</u> Tính hai tích phan: I= $\iint_D xdxdy$,

$J = \iint_D x^2dxdy$ D la tam giac co cac ñænh la A(2;3), B(7;2), C(4;5).



Giai: Xem hình

I1 = $\int_{x=2}^{x=4} xdx \int_{y=\frac{-x}{5}+\frac{17}{5}}^{y=x+1} dy = \left(\frac{2}{5}x^3 - \frac{6}{5}x^2\right)_{x=2}^{x=4}$ =8

I2 = $\int_{x=4}^{x=7} xdx \int_{y=\frac{-x}{5}+\frac{17}{5}}^{y=-x+9} dy = \left(-\frac{4}{5}x^3 + \frac{14}{5}x^2\right)_{x=4}^{x=7}$ =18

Vay I=8+18=26

Caùch 2: Töø 2 bieåu thöùc I, J muoán tính, ñoù chính laø momen tónh, momen quaùn tính cuûa tam giaùc ñ/v truïc Oy. Neân ôû ñaây ta coù theå aùp duïng coâng thöùc: $\iint\limits_D x\,dxdy = S_{Oy} = \frac{1}{6}\sum_{i=1}^{i=3}\left(x_i y_{i+1} - x_{i+1}y_i\right)\left(x_i + x_{i+1}\right)$

TT	1	2	3	4
x	2	7	4	2
y	3	2	5	3

Xem caùch baám maùy nhanh Vinacal-570MS (cuøng taùc giaû) cho coâng thöùc treân, ta coù

I= $\iint\limits_D x\,dxdy = \frac{1}{6}156 = 26$, töông töï: $\iint\limits_D y\,dxdy = \frac{1}{6}120 = 20$.

Tính $J = \iint\limits_D x^2 dxdy$. baèng caùch taùch 2 tích phaân:

$$J_1 = \int_{x=2}^{x=4} x^2 dx \int_{y=\frac{-x}{5}+\frac{17}{5}}^{y=x+1} dy = \int_{x=2}^{x=4} x^2\left(\frac{6x-12}{5}\right)dx = \frac{136}{5}$$

$$J_2 = \int_{x=4}^{x=7} x^2 dx \int_{y=\frac{-x}{5}+\frac{17}{5}}^{y=-x+9} dy = \int_{x=4}^{x=7} x^2\left(\frac{-4x+28}{5}\right)dx = \frac{459}{5} \Rightarrow J = \frac{136}{5} + \frac{459}{5} =$$

=119

Caùch 2: Nhôø coâng thöùc momen quaùn tính:

$$J = \iint\limits_D x^2 dxdy = J_{Oy} = \frac{1}{12}\sum_{i=1}^{i=n}\left(x_i y_{i+1} - x_{i+1}y_i\right)\left[\left(x_i + x_{i+1}\right)^2 - x_i x_{i+1}\right]$$

Baám maùy toång treân: $J_{Oy} = \frac{1}{12}1428 = 119$, töông töï: $J_{Ox} = \frac{1}{12}828 = 69$ ¢

Phaàn 2:

Ma traän +Heä phöông trình, Ñònh thöùc, KGVT.

CHÖÔNG 1
MA TRAÄN-HEÄ PHÖÔNG TRÌNH TUYEÁN TÍNH
BAØI 1: MA TRAÄN

1.1 Ñònh nghóa:

Moät ma traän m×n laø moät baûng chöõ nhaät goàm m×n phaàn töû nhö sau:

$$A=\begin{bmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \cdots\cdots & & \\ a_{m1} & a_{m2}\cdots & a_{mn} \end{bmatrix} \in M_{m,n}(\mathrm{R})$$

Ta coù caùc kyù hieäu: A=[a_{ij}] i = 1, ..., m, j = 1, ..., n

- Hai ma traän A vaø B baèng nhau, ghi laø A = B, neáu chuùng coù cuøng daïng kích thöôùc vaø caùc phaàn töû töông öùng baèng nhau.
- Trong tröôøng hôïp m=n goïi laø ma traän vuoâng, vaø neáu trong ma traän vuoâng maø coù taát caû phaàn töû ñeàu baèng 0, ngoaïi tröø caùc phaàn töû treân ñöôøng cheùo chính baèng 1, thì ta goïi ñoù laø ma traän ñôn vò.
- Ma traän chuyeån vò A^T ñöôïc xaây döïng töø ma traän A, baèng caùch ñoåi haøng thaønh coät cuûa A.

$$I_n=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix},\ A^T=\begin{bmatrix} a_{11} & a_{21}\cdots & a_{m1} \\ a_{12} & a_{22}\cdots & a_{m2} \\ \cdots\cdots & & \\ a_{1n} & a_{2n}\cdots & a_{m,n} \end{bmatrix} \in M_{n,m}(\mathrm{R})$$

- Ma traän tam giaùc treân, döôùi:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{21} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn} \end{vmatrix},\ \begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

<u>*Ví duï 1*</u>: Tìm x,y,z,w sao cho: $\begin{bmatrix} x+y & 2z+w \\ x-y & z-w \end{bmatrix}=\begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$

Ta ñöôïc heä phöông trình:

x + y = 3; x – y = 1; 2z + w = 5; z - w = 4

Nghieäm cuûa heä laø x = 2, y = 1, z = 3, w = -1.

<u>*Ví duï 2*</u>: hieån nhieân ta coù $\left(A^T\right)^T=A$

1.2 Caùc pheùp toaùn treân ñaïi soá ma traän:

Cho A, B,C, 0 ∈ $M_{m,n}$, α, β ∈ R

- Coäng: C=A+B ⇔ [c_{ij}]= [a_{ij}]+[b_{ij}]
- Nhaân vôùi höôùng: α A = [α a_{ij}] ∈ $M_{m,n}$, α ∈ R.
- Veát cuûa ma traän vuoâng $A \in M_n$, $tr(A)=\sum_{i=1}^{n} a_{ii}$

Ta coù caùc tính chaát sau:

(I) (A + B) + C =A + (B + C)
(II) A + 0 = A
(III) A + (-A) = 0
(IV) A + B = B + A
(V) α (A + B) = α A + α B
(VI) (α + β) A = αA + βA
(VII) (α. β) A = α (β A)
(VIII) 1. A = A

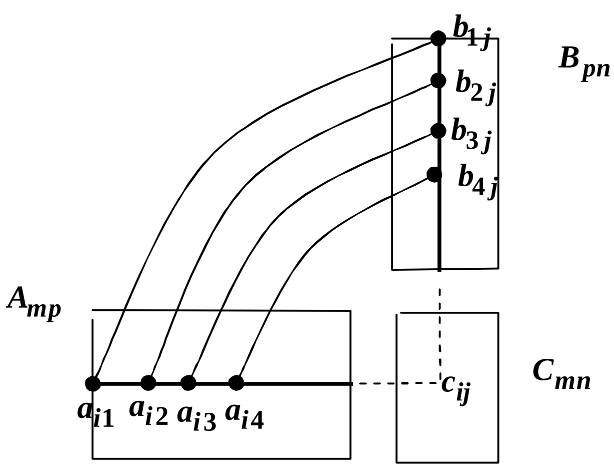
Vay tap cac ma tran ∈ $M_{m,n}(R)$ co he so thöc, la mot KGVT tren R.

- Nhan ma tran vôi ma tran:

Gia sö $[a_{ij}]$ va $[b_{ij}]$ la cac ma tran sao cho so cot cua A thì bang so dong cua B nghóa la $A_{m,p}$ va $B_{p,n}$. Khi ño tích cua ma tran A va B, ghi la AB la ma tran m x n ma so hang c_{ij} thu ñöôc bang cach nhan vo höông veto dong thö i cua A vôi veto cot thö j cua B, cu the:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Sô ño nhan ma tran:



Ví du1: Sô ño nhan ba ma tran:

[A(2,2)* B(2,3)]*C(3,2)= D(2,2) nhö hình ben döôi:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

					0	2
		1	1	−1	1	1
		2	−1	1	−1	−1
1	−1	−1	2	−2	4	2
2	1	4	1	−1	2	10

,

ma tran cuoi D(2,2)=$\begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix}$.

Ta co the viet ñoan ma Pascal tích 3 ma tran

[A(m,n)* B(n,p)]*C(p,q)= D(m,q) nhö sau:

Khai bao dimension A(m,n), B(n,p), C(p,q), D(m,q)

for i=$\overline{\mathbf{1,}\boldsymbol{m}}$ do begin

for j=$\overline{\mathbf{1,}\boldsymbol{q}}$ do begin

D(i,j)=0;

for k=$\overline{\mathbf{1,}\boldsymbol{p}}$ do begin

for kk=$\overline{\mathbf{1,}\boldsymbol{n}}$ do begin

D(i,j)= D(i,j)+ A(i,kk) * B(kk,k) * C(k,j);

end

end

end

end

<u>*Ví du 2:*</u> $\boldsymbol{C}=\begin{bmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\neq \mathbf{0}, \boldsymbol{D}=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{1} & \mathbf{0}\end{bmatrix}\neq \mathbf{0}\Rightarrow \boldsymbol{CD}=\mathbf{0}=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \boldsymbol{DC}=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{1} & \mathbf{0}\end{bmatrix}$

Tö ñay ta thay rang, trong ma tran neu co CD=0 $\not\Rightarrow$ (C=0 hay D=0).

Ví du khac $\boldsymbol{C}=\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{2}\\ \mathbf{0} & -\mathbf{2} & \mathbf{1}\\ \mathbf{0} & \mathbf{1} & -\mathbf{1}\end{bmatrix}\neq \mathbf{0}, \boldsymbol{D}=\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{4}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\neq \mathbf{0}\Rightarrow \boldsymbol{CD}=\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}.$

<u>*Ví du3:*</u> Ñac biet nhan 2 ma tran hang va cot, ket qua la so thöc:

$$\begin{bmatrix}h_1 & h_2 & \cdots & h_n\end{bmatrix}\begin{bmatrix}c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix}=h_1c_1+h_2c_2+h_3c_3+\cdots+h_nc_n\in \boldsymbol{R}$$

giong nhö tích vo höông chính tac ña biet ô lôp 12 pho thong.

<u>*Tính chat tích ma tran:*</u>

1/ $AI_n = I_n A$ vôi $A \in M_{n,n}$, $I_n \in M_{n,n}$ (ma tran ñôn vò cap n)

2/ A(BC)=(AB)C

3/ A(B+C)=AB+AC, (B+C)A=BA+CA

4/ Tong quat ta khong co AB=BA.

<u>*Ví du 4*</u>: Lay lai ví du tren ñe kiem tra sô ño nhan A(BC)

$$\boldsymbol{A}=\begin{bmatrix}\mathbf{1} & -\mathbf{1}\\ \mathbf{2} & \mathbf{1}\end{bmatrix}, \boldsymbol{B}=\begin{bmatrix}\mathbf{1} & \mathbf{1} & -\mathbf{1}\\ \mathbf{2} & -\mathbf{1} & \mathbf{1}\end{bmatrix}, \boldsymbol{C}=\begin{bmatrix}\mathbf{0} & \mathbf{2}\\ \mathbf{1} & \mathbf{1}\\ -\mathbf{1} & -\mathbf{1}\end{bmatrix},$$

			0	2
			1	1
			–1	–1
1	1	–1	2	4
2	–1	1	–2	2
	1	–1	4	2
	2	1	2	10

ma tran cuoi A(BC)= D(2,2) = $\begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix}$ so sanh vôi ví du 1.

- Luy thöa ma tran: A $\in$ M_n(R), ta ky hieu nhö sau:

$A^0 = I$, $A^1 = A$, $A^2 = AA$, $A^m = A^{m-1}A$.

<u>*Ví du 5*</u>: $A = \begin{bmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

<u>*Ví du 6*</u>: Cho ma trận A vuong cấp 2007 ma phan tử ở dong thứ i la $(-1)^i i$, thì phần tử ở ô dong 2 cot 3 cua A^2 la:

$$2\sum_{1}^{2007}(-1)^i i = 2\left[(-1+2)+(-3+4)-........(-2005+2006)-2007\right] =$$

$$= 2\left[\frac{2006}{2}\times(1)-2007\right] = -2008$$

<u>*Ñònh ly:*</u> Gia sö A, B $\in$ M_n(R), va giao hoan ñöôc, nghia la AB=BA. Khi ño:

1/ $(AB)^m = A^m B^m$.

2/ $A^m - B^m = (A-B)\left(A^{m-1} + A^{m-2}B + A^{m-3}B^2 + \cdots + B^{m-1}\right)$.

3/ $(A+B)^m = \sum_{i=0}^{i=m} C_m^i A^{m-i} B^i$. $C_m^i = \dfrac{m!}{i!(m-i)!}$

<u>*Ñònh ly:*</u> A^T la ma tran ñöôc thiet lap tö A bang cach ñoi dong thanh cot vay: A $\in$ $M_{m,n}$(R), va A^T $\in$ $M_{n,m}$(R)). Ta co cac tính chat sau:

i/ $\left(A^T\right)^T = A$.

ii/ $(A+B)^T = A^T + B^T$.

iii/ $(AB)^T = B^T A^T$.

BAI 2: HE PHÖÔNG TRÌNH TUYEN TÍNH

2.1 Ñònh nghóa:

He phöông trình tuyen tính la he phöông trình co dang tong quat (m $\neq$ n):

$$(1)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow A_{m\times n}.x = b$$

Trong ño:

(2) $A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & L & a_{1n} \\ a_{21} & a_{22} & L & a_{2n} \\ L & O & O & L \\ L & O & O & L \\ a_{m1} & a_{m2} & L & a_{mn} \end{bmatrix}$ la ma tran he so cua (1),

$x_{n\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ L \\ x_n \end{bmatrix}$; $b_{m\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ L \\ b_m \end{bmatrix}$

(3) $A_{m\times(n+1)} = \left[\begin{array}{ccc|c} a_{11} & L & a_{a1} & b_1 \\ L & L & L & L \\ a_{m1} & L & a_{mn} & b_m \end{array}\right] = \left[A \middle| b\right] = \left[A'\right]$,

la ma tran môi rong cua he (1).

He nay co m phöông trình tuyen tính, n an so x_1, ..., x_n.

He ñöôc goi la <u>thuan nhat</u> neu cac hang so b_1, ..., b_m tat ca bang 0.

2.2 Nghiem cua he phöông trình tuyen tính:

- Bo (x_1, x_2, x_3,.....x_n) goi la nghiem neu chung thoa man he (1). Giai he (1) la tìm tat ca bo nghiem tren.
- Hai he phöông trình tuyen tính goi la töông ñöông neu chung cung chung mot tap nghiem.
- Chu y he thuan nhat luon co ít nhat mot nghiem tam thöông la (0,0,...,0).

2.3 Cac phep bien ñoi sô cap (PBÑSC) *tren dong cua ma tran:*

Phep bien ñoi sô cap tren dong (PBÑSC) cua ma tran gom:

1/ Ñoi cho 2 dong cho nhau.

2/ Nhan mot dong vôi mot he so khac khong.

3/ Cong vao mot dong vôi mot dong khac sau khi ña nhan vôi mot hang so.

A ∈ $M_{m,n}$(R). Mot PBÑSC e tren ma tran A ñe ñöôc ma tran $A^/$, ta ky hieu $A \xrightarrow{e} A^/$.

Nhan xet: Neu e la PBÑSC bien A thanh $A^/$, thì cung co PBÑSC $e^/$ ñe sao cho $A^/ \xrightarrow{e^/} A$.

- <u>*Ñònh nghia:*</u> A, B ∈ $M_{m,n}$(R) ñöôc goi la töông ñöông dong, khi A co ñöôc tö B qua höu han mot so PBÑSC. Luc ño ta kí hieu A ~ B.
- <u>*Ñònh nghia:*</u> Mot ma tran vuong S ñöôc goi la ma tran sô cap khi co mot PBÑSC sao cho $I \xrightarrow{e} S$. Khi ño ta viet S = e(I).

<u>*Ví du1:*</u> S= $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ la ma tran sô cap vì $I \xrightarrow{d_1+2d_3} S$

- <u>*Menh ñe:*</u> A ∈ $M_{m,n}$(R) (<u>khong can vuong</u>). e la Mot PBÑSC. Khi ño:

$A \xrightarrow{e} A^/ \in$ $M_{m,n}$ va $I_m \xrightarrow{e} S \in M_{m\times m}$ thì: $A^/$ =SA.

<u>*Ví du2:*</u> A= $\begin{bmatrix} 1 & 1 & 2 \\ 4 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, $A \xrightarrow{d_1+2d_3} A^/ = \begin{bmatrix} 3 & 5 & 8 \\ 4 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$S_1=\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}$ laø ma traän sô caáp vì $I \xrightarrow{d_1+2d_3} S_1$

$A^/=\begin{bmatrix}3&5&8\\4&3&2\\1&2&3\end{bmatrix}=S_1A=\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&2\\4&3&2\\1&2&3\end{bmatrix}$. Tieáp tuïc, hoaùn vò doøng 2, 3:

$$A^/=\begin{bmatrix}3&5&8\\4&3&2\\1&2&3\end{bmatrix}\xrightarrow{d_2\leftrightarrow d_3}A^{//}=\begin{bmatrix}3&5&8\\1&2&3\\4&3&2\end{bmatrix},\ I\xrightarrow{d_2\leftrightarrow d_3}S_2$$

$S_2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$. Vaäy $A^{//}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&2\\4&3&2\\1&2&3\end{bmatrix}=S_2S_1A$

<u>*Ví duï3:*</u> $A=\begin{bmatrix}1&1&2\\1&2&3\end{bmatrix}$, $A\xrightarrow{d_1+2d_3}A^/=\begin{bmatrix}3&5&8\\1&2&3\end{bmatrix}$

$S_1=\begin{bmatrix}1&2\\0&1\end{bmatrix}$ laø ma traän sô caáp vì $I\xrightarrow{d_1+2d_3}S_1$. Haõy thöû laïi:

$A^/=\begin{bmatrix}3&5&8\\1&2&3\end{bmatrix}=S_1A=\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}1&1&2\\1&2&3\end{bmatrix}$

• <u>*Heä quaû:*</u> A, B $\in$ $M_{m,n}$(R). A ~ B khi vaø chæ khi toàn taïi caùc ma traän sô caáp S_1, S_2, S_3,...., S_k caáp m sao cho: B= S_k S_{k-1}.... S_2. S_1. A

Do $A\xrightarrow{e_1}A_1\xrightarrow{e_2}A_2\xrightarrow{e_3}A_3 \ldots \xrightarrow{e_k}A_k=B$

ta coù $A_1 = S_1A$, $A_2 = S_2A_1$, $A_3 = S_3A_2$,..... . . ., $B=A_k = S_kA_{k-1}$,
suy ra ñpcm.

• <u>*Chuù yù:*</u> Neáu e laø PBÑSC treân coät
A $\in$ $M_{m,n}$(R) (<u>khoâng caàn vuoâng</u>). e laø Moät PBÑSC treân coät. Khi ñoù:

$A\xrightarrow{e}A^/\in M_{m,n}$ vaø $I_n\xrightarrow{e}S\in M_{n\times n}$ thì: $A^/=AS$.

<u>Ví duï:</u> $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}\xrightarrow{-1c2+c1}A^/=\begin{bmatrix}-1&2\\-1&4\end{bmatrix}$

$$I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}\xrightarrow{-1c2+c1}S=\begin{bmatrix}1&0\\-1&1\end{bmatrix}.$$

Haõy thöû laïi: $A^/=\begin{bmatrix}-1&2\\-1&4\end{bmatrix}=A.S=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}$

2.4 Ñònh lyù veà caùc pheùp bieán ñoåi sô caáp:

<u>*Ñònh lyù:*</u> Sau höõu haïn caùc pheùp bieán ñoåi sô caáp treân heä (1) ta thu ñöôïc heä töông ñöông.
Chöùng minh ñònh lyù naøy khaù ñôn giaûn.

<u>*Ví duï1:*</u>

Heä $\begin{cases}x_1-3x_2-x_3+2x_4=3\\3x_1+5x_2+9x_3-4x_4=-8,\\4x_1-3x_2+5x_3+7x_4=14\end{cases}$ $\begin{bmatrix}1&-3&-1&2&3\\3&5&9&-4&-8\\4&-3&5&7&14\end{bmatrix}\to \ldots \to$

$$\begin{bmatrix} 1 & -3 & -1 & 2 & 3 \\ 0 & 9 & 9 & -1 & 2 \\ 0 & 14 & 12 & -10 & -17 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -1 & 2 & 3 \\ 0 & 1 & 1 & \frac{-1}{9} & \frac{2}{9} \\ 0 & 0 & -2 & \frac{-76}{9} & \frac{-181}{9} \end{bmatrix}$$

Vay he ñau töông ñöông: $\begin{cases} x_1 - 3x_2 - x_3 + 2x_4 = 3 \\ 9x_2 - 9x_3 - x_4 = 2 \\ -2x_3 - \frac{76}{9}x_4 = -\frac{181}{9} \end{cases}$

va ta tìm nghiem tren he ñôn gian nay de dang hôn nhieu.

Vay sau höu han cac phep bien ñoi sô cap tren tran $A_{m,,n}$, ta thu ñöôc ma tran töông ñöông, ma dang cua ma tran sau cung $A^{/}$ co dang bac thang (nhöng he so khac zero tao thanh bac thang rut gon).

Ñònh nghóa ma tran bac thang: $R \in M_{m,,n}(R)$ ñöôc goi la ma tran bac thang rut gon neu cac ñieu kien sau ñay ñöôc thoa:

i/ Cac dong zero (neu co) phai ô ben döôi cac dong khac zero (neu co).

ii/ Phan tö ñau tien khac 0 tren cac dong khac zero (neu co) la so 1, va ben döôi cot cua so mot nay tat ca ñeu bang 0.

Ví du: ma tran bac thang co dang nhö sau: $\begin{bmatrix} \boxed{1} & * & * & \cdots & * \\ 0 & \boxed{1} & * & \cdots & * \\ 0 & 0 & \boxed{1} & \cdots & * \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$

Ñònh ly: Moi ma tran ñeu töông ñöông dong vôi ma tran co dang bac thang rut gon duy nhat.

Ví du2: A= $\begin{bmatrix} 0 & 2 & -4 \\ -1 & -4 & 5 \\ 3 & 1 & 7 \\ 0 & 5 & -10 \\ 2 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -4 & 5 \\ 0 & 2 & -4 \\ 0 & 5 & -10 \\ 0 & -11 & 22 \\ 0 & -5 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ =B

B la la dang ma tran bac thang rut gon töông ñöông vôi A.

Bay giô neu goi x, y, z la nghiem cua $\begin{cases} 0x + 2y - 4z = 0 \\ -1x - 4y + 5z = 0 \\ 3x + 1y + 7z = 0 \\ 0x + 5y - 10z = 0 \\ 2x + 3y + 0z = 0 \end{cases}$ thì chung cung la nghiem cua $\begin{cases} 1x + 3z = 0 \\ 1y - 2z = 0 \end{cases}$ va nghiem nay la: $\begin{cases} x = -3t \\ y = 2t \\ z = t \end{cases}$.

Luc nay ta ñònh nghóa hang ma tran nhö sau:

Ñònh nghóa haïng cuûa ma traän: Soá haøng khaùc zero cuûa ma traän daïng baäc thang ruùt goïn cuûa ma traän A ñöôïc goïi laø haïng cuûa ma traän A. Kyù hieäu laø r(A), ñoâi khi coøn kí hieäu laø rank(A).

Ngoaøi ta coøn ñònh nghóa haïng cuûa ma traän laø baäc lôùn nhaát cuûa moät ñònh thöùc con naøo ñoù khaùc zero (ñöôïc trích ra töø A, k haøng, k coät baát kyø vaø moïi ñònh thöùc caáp k+1 trôû ñi ñeàu baèng 0). Ta coù theå chöùng minh hai ñònh nghóa naøy laø töông ñuông.

Ví duï3: A= $\begin{bmatrix} 2 & 1 & 4 & 5 & 6 \\ 0 & 3 & 2 & 3 & 0 \\ 0 & 0 & 4 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, coù haïng laø 3, vì ñaây laø ma traän baäc thang chöùa ruùt goïn, vaø moïi ma traän baäc 4 ñeàu baèng 0, vì chöùa haøng thöù tö toaøn baèng 0. Ví duï naøy cho ta thaáy 2 ñònh nghóa veà haïng ôû treân laø töông ñöông.

Meänh ñeà: A $\in M_{m,,n}$(R) khi ñoù:

i/ $0 \le$ r(A) $\le$ min (m,n).

ii/ Neáu A~B thì r(A) =r(B).

iii/ r(A) =0 <==> A=0.

iv/ r(A)=r(A^T)

2.5 Ñònh lyù Kronecker-Capelli:

Ñònh lyù: Heä $A_{m\times n}.x = b$ (1), trong ñoù: $[A'] = [A|\,b]$, ta coù caùc keát luaän:

a/ Ta coù hoaëc r(A) = r($A^/$) hoaëc r($A^/$)= r(A) +1.

b/ (1) coù nghieäm khi vaø chæ khi r(A)= r($A^/$), cuï theå:

- Neáu r(A)= r($A^/$)= n, (n laø soá aån) thì heä coù nghieäm duy nhaát.
- Coøn neáu r(A)= r($A^/$) < n, (n laø soá aån) thì heä coù voâ soá nghieäm.

Vôùi soá aån töï do laø n - r(A)

c/ r(A) < r($A^/$)= r(A) +1, heä voâ nghieäm.

Ví duï1: Döïa vaøo ñònh lyù Kronecker-Capelli, xeùt vò trí töông ñoái cuûa 4 maët phaúng trong R^3: chæ roõ caáu truùc nghieäm cuûa heä phöông trình neáu coù:

$$\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array} \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \\ a_3x + b_3y + c_3z + d_3 = 0 \\ a_4x + b_4y + c_4z + d_4 = 0 \end{cases},$$

ñaët A= $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$, $[A\,|\,b] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & -d_1 \\ a_2 & b_2 & c_2 & -d_2 \\ a_3 & b_3 & c_3 & -d_3 \\ a_4 & b_4 & c_4 & -d_4 \end{array}\right]$

Do soá aån ít hôn soá phöông trình neân khoâng bao giôø coù r(A)= r([A|b])=4. Vaäy chæ coøn caùc tröôøng hôïp sau:

1/ r(A)= r([A|b])=3, $\exists$! nghieäm laø moät ñieåm (x_0, y_0, z_0).

2/ r(A)=3 < r([A|b])=4, voâ nghieäm. Heä suy bieán thaønh ba maët phaúng (mp) caét nhau, vaø coù mp (4) // vôùi moät trong ba mp treân.

3/ r(A)=2 = r([A|b])=2, voâ soá nghieäm. Heä suy bieán thaønh hai maët phaúng caét nhau. Nghieäm naèm treân ñöôøng thaúng giao tuyeán.

4/ r(A)=2 < r([A|b])=3, voâ nghieäm. Heä suy bieán thaønh hai maët phaúng caét nhau, vaø coù moät mp // vôùi moät trong hai mp treân.

5/ r(A)=2 = r([A|b])=4, voâ nghieäm. Heä suy bieán thaønh hai maët phaúng caét nhau, coøn hai mp // vôùi moät trong hai mp treân.

6/ r(A)=1 = r([A|b])=1, voâ soá nghieäm. Heä suy bieán thaønh moät maët duy nhaát, coøn ba mp kia truøng vôùi mp treân.

7/ r(A)=1 < r([A|b])=2, voâ nghieäm. Heä suy bieán thaønh moät maët duy nhaát, vaø coù moät mp // vôùi mp treân.

8/ r(A)=1 < r([A|b])=3, voâ nghieäm. Heä suy bieán thaønh moät maët duy nhaát, vaø coù hai mp // vôùi mp treân.

9/ r(A)=1 < r([A|b])=4, voâ nghieäm. Heä suy bieán thaønh moät maët duy nhaát, vaø coù ba mp // vôùi mp treân. ¢

Ví duï2: Giaûi heä: $\begin{cases} x_1-2x_2-x_3 \quad +x_4=2 \\ 3x_1-6x_2-4x_3+2x_4=4 \\ -x_1+2x_2+2x_3 \qquad =0 \end{cases}$ <==> Ax=b (1)

Ta xeùt caáp 3: $\begin{vmatrix} 1 & -2 & -1 \\ 3 & -6 & -4 \\ -1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -1 \\ 3 & -6 & -4 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = 0$

Xeùt ñònh thöùc caáp 2: $\begin{vmatrix} -2 & -1 \\ -6 & -4 \end{vmatrix} \neq 0$, neân rank(A)=2. Xeùt ma traän môû roäng A$'$ ta coù:

$\begin{vmatrix} 1 & -2 & 2 \\ 3 & -6 & 4 \\ -1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 2 \\ 3 & -6 & 4 \\ 0 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = 0$,

=> r(A$'$)=2=r. Vaäy heä coù nghieäm (coù 2= n -r = 4-2 baäc töï do)

$\begin{cases} x_1-x_3 \quad =2+2x_2-x_4 \\ 3x_1-4x_3=4+6x_2-2x_4 \end{cases}$ ==> $x_1=2x_2-2x_4+4$ $x_3=-x_4+2$,

Caùch 2: $\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 2 \\ 3 & -6 & -4 & 2 & 4 \\ -1 & 2 & 2 & 0 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ => keát quaû.

Nhaän xeùt: $(x_1,x_2,x_3,x_4) = \underbrace{(4,0,2,0)}_{x^{(0)}} + x_2\underbrace{(2,1,0,0)}_{x^{(1)}} + x_4\underbrace{(-2,0,-1,1)}_{x^{(2)}}$, ta coù $x^{(0)}$ laø N^0 cuûa Ax=b, coøn $x^{(1)}$, $x^{(2)}$ laø N^0 cuûa Ax=0, haõy kieåm chöùng!

Ví duï3: Giaûi vaø bieän luaän heä phöông trình theo tham soá m:

$$\begin{cases} x+y+z=m \\ x+y+mz=1 \\ x+my+z=1 \\ mx+y+z=1 \end{cases}$$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & m \\ 1 & 1 & m & 1 \\ 1 & m & 1 & 1 \\ m & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & m \\ 0 & 0 & m-1 & 1-m \\ 0 & m-1 & 0 & 1-m \\ 0 & 1-m & 1-m & 1-m^2 \end{array}\right] \to$ Neáu m-1=0 thì

$$\to \left[\begin{array}{ccc|c}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{array}\right] \to \text{He co vo so nghiem}$$

Neu m-1 $\neq$ 0 thì tiep tuc bien ñoi.

$$\to \left[\begin{array}{ccc|c}1&1&1&m\\0&1&0&-1\\0&0&1&-1\\0&1&1&1+m\end{array}\right] \ldots \to \left[\begin{array}{ccc|c}1&0&1&m+1\\0&1&0&-1\\0&0&1&-1\\0&0&0&m+3\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&0&m+2\\0&1&0&-1\\0&0&1&-1\\0&0&0&m+3\end{array}\right]$$

Tom lai:
- m $\neq$ 1 va m $\neq$ -3 he vo nghiem.
- m = -3 he duy nhat nghiem x = y =z = -1.
- m =1 he vo so nghiem.

Ví du4: Giai va bien luan he phöông trình theo cac thong so a,b:

$$\begin{cases} bx & +ay & +z & =1 \\ (2b-1)x+ay & & +3z & =1 \\ bx & +ay & +(b+3)z & =1 \end{cases}$$

Giai:

$$\begin{array}{cccc} z & y & x & \\ \end{array}$$
$$\begin{bmatrix} 1 & a & b & 1 \\ 3 & a & 2b-1 & 1 \\ b+3 & a & b & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & a & b & 1 \\ 0 & -2a & -b-1 & -2 \\ 0 & a(-b-2) & -b(b+2) & -b-2 \end{bmatrix}$$

a/ b $\neq$ -2: => $\begin{bmatrix} 1 & a & b & 1 \\ 0 & 2a & b+1 & 2 \\ 0 & -a & -b & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & a & b & 1 \\ 0 & -a & -b & -1 \\ 0 & 0 & -b+1 & 0 \end{bmatrix}$

=> $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 1 \\ 0 & 0 & -b+1 & 0 \end{bmatrix}$

b $\neq$ 1, a $\neq$ 0: => N^0 = (0, 1/a, 0) duy nhat nghiem.

$$\begin{array}{ccc} z & y & x \end{array} \qquad\qquad \begin{array}{ccc} z & x & y \end{array}$$
b =1 => $\begin{bmatrix} 1 & a & 1 & 1 \\ 0 & -a & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, r(A)=2=r(A$^/$)<n=3

==>Vo so N^0, khong gian nghiem la ñöông thang.

$$\begin{array}{ccc} z & y & x \end{array} \qquad\qquad \begin{array}{ccc} z & x & y \end{array}$$
b/ b = -2: => $\begin{bmatrix} 1 & a & -2 & 1 \\ 0 & 2a & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & a & 1 \\ 0 & -1 & 2a & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, r(A)=2=r(A$^/$)<n=3

==>Vo so N^0, khong gian nghiem la ñöông thang.

Tom lai:
- b $\neq$ 1, b $\neq$ -2, a $\neq$ 0: => N^0 = (0, 1/a, 0) duy nhat nghiem.
- b $\neq$ 1, b $\neq$ -2, a = 0: => Vo N^0.

• b = 1, Vôùi soá N^0, khoâng gian nghieäm laø ñöôøng thaúng..
• b = -2, Vôùi soá N^0, khoâng gian nghieäm laø ñöôøng thaúng.

Thaät ra ta coù $2^3 = 8$ tröôøng hôïp, ruùt veà $\to 6$ tröôøng hôïp sau, nhöng thu veà chæ coù 4 tröôøng hôïp toùm taét ôû treân:

1/ $a = 0, b \neq 1, b \neq -2$

2/ -- $b = 1$

3/ -- $b = -2$

4/ $a \neq 0, b \neq 1, b \neq -2$

5/ -- $b = 1$

6/ -- $b = -2$

<u>*Caùch khaùc:*</u> $A = \begin{bmatrix} b & a & 1 \\ 2b-1 & a & 3 \\ b & a & b+3 \end{bmatrix}$ (xem chöông 2, tính ñònh thöùc)

$$\begin{vmatrix} b & a & 1 \\ 2b-1 & a & 3 \\ b & a & b+3 \end{vmatrix} = \begin{vmatrix} b & a & 1 \\ -b-1 & -2a & 0 \\ -2b-b^2 & -2a-ab & 0 \end{vmatrix} = a\begin{vmatrix} -b-1 & -2 \\ -2b-b^2 & -2-b \end{vmatrix}$$

$\Rightarrow \det(A) = -b^2a - ab + 2a = -a(b+2)(b-1)$. Ta theá caùc giaù trò b=1, b= -2, a=0 vaøo phöông trình vaø lyù luaän töông töï nhö treân ta seõ coù keát quaû töông töï.

<u>*Ví duï5*</u>: Giaûi, bieän luaän heä phöông trình theo tham soá λ: $\begin{cases} \lambda x + y + z = 1 \\ x + \lambda y + z = \lambda \\ x + y + \lambda z = \lambda^2 \end{cases}$

Giaûi:

$$\begin{bmatrix} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & \lambda \\ 1 & 1 & \lambda & \lambda^2 \end{bmatrix} \to \begin{bmatrix} 1 & \lambda & 1 & \lambda \\ \lambda & 1 & 1 & 1 \\ 1 & 1 & \lambda & \lambda^2 \end{bmatrix} \text{-->} \begin{bmatrix} 1 & \lambda & 1 & \lambda \\ 0 & -\lambda^2+1 & 1-\lambda & 1-\lambda^2 \\ 0 & 1-\lambda & \lambda-1 & \lambda^2-\lambda \end{bmatrix}$$ a/ λ ≠ 1:

$$\begin{bmatrix} 1 & \lambda & 1 & \lambda \\ 0 & \lambda+1 & 1 & 1+\lambda \\ 0 & 1 & -1 & -\lambda \end{bmatrix} \text{-->} \begin{bmatrix} 1 & \lambda & 1 & \lambda \\ 0 & 1 & -1 & -\lambda \\ 0 & 1+\lambda & 1 & 1+\lambda \end{bmatrix} \text{-->}$$

$$\text{-->} \begin{bmatrix} 1 & \lambda & 1 & \lambda \\ 0 & 1 & -1 & -\lambda \\ 0 & 0 & \lambda+2 & (\lambda+1)^2 \end{bmatrix}$$

vaø λ ≠ -2: => N^0 duy nhaát nghieäm: $\left\{ x = -\dfrac{\lambda+1}{\lambda+2}, y = \dfrac{1}{\lambda+2}, z = \dfrac{(\lambda+1)^2}{\lambda+2} \right\}$

λ = -2: --> Voâ N^0

b/ λ = 1:--> $\begin{bmatrix} 1 & \lambda & 1 & \lambda \\ 0 & -\lambda^2+1 & 1-\lambda & 1-\lambda^2 \\ 0 & 1-\lambda & \lambda-1 & \lambda^2-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ N^0 laø mp.

Cach khac: $A=\begin{bmatrix} l & 1 & 1\\ 1 & l & 1\\ 1 & 1 & l\end{bmatrix} \Rightarrow \det(A) = l^3 - 3l + 2 = (l-1)^2(l+2)$

• λ = 1:--> $\begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ co N^0 la mp.

• λ = -2:--> $\left[\begin{array}{ccc|c} 1 & -2 & 1 & -2\\ -2 & 1 & 1 & 1\\ 1 & 1 & -2 & 4\end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & -2\\ 0 & -3 & 3 & -3\\ 0 & 3 & -3 & 6\end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & -2\\ 0 & -3 & 3 & -3\\ 0 & 0 & 0 & 3\end{array}\right]$

--> Vo N^0.

• λ ≠ -2 va λ ≠ 1, duy nhat nghiem.

2.6 Phöông phap khö Gauss:

Ñe giai he phöông trình tuyen tính, ta dung cac phep bien ñoi sô cap, ñöa he ve mot he môi töông ñöông co dang bac thang, sau ño dung ñình ly Cronecker-Capelli ñe xem he co nghiem hay khong? Neu co ta tìm tap nghiem bang cach the tö döôi len tren ñe giai.

Ví du 1: Giai he phöông trình:

$\begin{cases} x_1 + 2x_2 + x_3 = 1\\ 2x_1 + 5x_2 + x_3 = 6\\ -x_1 + 4x_2 + 2x_3 = 2\end{cases}$, ap phep bien ñoi sô cap vao, ta co:

$$\begin{cases} x_1 + 2x_2 + x_3 = 1\\ 2x_1 + 5x_2 + x_3 = 6\\ -x_1 + 4x_2 + 2x_3 = 2\end{cases} \rightarrow \begin{cases} x_1 + 2x_2 + x_3 = 1\\ \quad x_2 - x_3 = 4\\ \quad +6x_2 + 3x_3 = 3\end{cases} \rightarrow \begin{cases} x_1 + 2x_2 + x_3 = 1\\ \quad x_2 - x_3 = 4\\ \quad\quad +9x_3 = -21\end{cases}$$

Ta thay ngay r(A)= r(A$^/$)=r=3=n (co duy nhat nghiem),

the tö döôi len tren ta co he duy nhat nghiem: x=$\left(0; \frac{5}{3}; \frac{-7}{3}\right)$.

Trong qua trình bien ñoi ta chæ tính toan tren cac he so nen bat ñau tö ñay, ñe tien ta chæ bieu dien ma tran môû rong cho cac phep bien ñoi sô cap ma thoi.

Ví du 2: Giai he phöông trình:

$$\begin{cases} x_1 + 2x_2 - x_3 = 3\\ 4x_1 + x_2 + 2x_3 = 1\\ 2x_1 - 3x_2 + 4x_3 = 2\end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3\\ 4 & 1 & 2 & 1\\ 2 & -3 & 4 & 2\end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3\\ 0 & -7 & 6 & -11\\ 0 & -7 & 6 & 6\end{array}\right] \rightarrow$$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3\\ 0 & -7 & 6 & 11\\ 0 & 0 & 0 & 7\end{array}\right] \Rightarrow$ he tren vo nghiem r(A)=2 < r(A$^/$)=3. ¢

Ví du 3: Giai he phöông trình: $\begin{cases} x_1 + x_2 + x_3 = 6\\ 2x_1 - x_2 + x_3 = 3\\ x_1 - x_2 + 2x_3 = 5\\ 3x_1 - 6x_2 + 5x_3 = 6\end{cases}$

Ta bien ñoi: $\left[\begin{array}{ccc|c}1&1&1&6\\2&-1&1&3\\1&-1&2&5\\3&-6&5&6\end{array}\right]\to\left[\begin{array}{ccc|c}1&1&1&6\\0&-3&-1&-9\\0&-2&1&-1\\0&-9&2&-12\end{array}\right]$, ñoi vò trí x_2 va x_3:

$$\begin{array}{c}\begin{array}{ccc}\mathbf{x_1}&\mathbf{x_3}&\mathbf{x_2}\end{array}\\ \to\left[\begin{array}{ccc|c}1&1&1&6\\0&-1&-3&-9\\0&1&-2&-1\\0&2&-9&-12\end{array}\right]\end{array}\to\left[\begin{array}{ccc|c}1&1&1&6\\0&-1&-3&-9\\0&0&-5&-10\\0&0&-15&-30\end{array}\right]\to\begin{array}{c}\begin{array}{ccc}\mathbf{x_1}&\mathbf{x_3}&\mathbf{x_2}\end{array}\\ \left[\begin{array}{ccc|c}1&1&1&6\\0&-1&-3&-9\\0&0&1&2\\0&0&0&0\end{array}\right]\end{array}$$

Ta thay ngay r(A)= r(A$^{/}$)=r=3 = n=3 (co duy nhat nghiem), x_2 =2,

$$\left[\begin{array}{ccc|c}1&1&1&6\\0&-1&-3&-9\\0&0&1&2\\0&0&0&0\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&-2&-3\\0&-1&-3&-9\\0&0&1&2\\0&0&0&0\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&0&1\\0&-1&0&-3\\0&0&1&2\\0&0&0&0\end{array}\right]$$

==> x_3 =3, x_1 =1,

Ví du 4: Cho he: Ax=b <==> $\begin{bmatrix}1&1&1&1&1\\3&2&1&1&-3\\0&-1&2&-1&2\\3&1&-1&-1&-9\\-6&-7&4&-5&12\end{bmatrix}.x=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\\b_5\end{bmatrix}$ (1)

Tìm ñieu kien cua b ñe (1) co nghiem, tìm cau truc nghiem cua (1). Sinh vien hay kiem chöng ket qua sau: cau truc nghiem la

$$\left(x_1,x_2,x_3,x_4,x_5\right)=\underbrace{\left(b_3+b_1,\ b_2-2b_3-3b_1,\ 0,\ b_3-b_2+3b_1,\ 0\right)}_{x^{(0)}}+$$

$+x_4\underbrace{\left(-3,6,1,-4,0\right)}_{x^{(1)}}+x_5\underbrace{\left(3,-2,-2,0,1\right)}_{x^{(2)}}$, trong ño: $b_4=-3b_1+2b_2$, $b_5=3b_3-2b_2$, ta co $x^{(0)}$ la N^0 rieng cua Ax=b, con $x_4x^{(1)}+x_5x^{(2)}$ la N^0 tong quat cua Ax=0, trong ño : $x^{(1)}$, $x^{(2)}$ la 2 N^0 rieng ÑLTT cua Ax=0, hay kiem chöng!

Chu y co the co nhieu ñap so bieu dien khac nhau. ¢

Ñònh ly: Tap bo (x_1, x_2, x_3,.....,x_n) cua he thuan nhat tao thanh mot khong gian vectô.

Chöng minh danh cho Hoc Sinh.

2.7 Nghiem cua he thuan nhat:

Xet he thuan nhat:

$$(4)\begin{cases}a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=0\\a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=0\\\ldots\ldots\ldots\ldots\ldots\ldots=0\\a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=0\end{cases}\Leftrightarrow A_{m\times n}.x=0=\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$$

Ñònh ly: Mot he thuan nhat cac phöông trình tuyen tính vôi so an nhieu hôn so phöông trình (m < n), co mot nghiem khac 0 (nghiem khong tam thöông). Khong gian nghiem la mot khong gian con cua R^n.

Do aùp duïng ñònh lyù Kronecker-Cappeli ta luoân coù r(A)= r(A/) < n.
Vì A/ =A ==> r(A)= r(A/) ≤ min (m,n) = m < n.

2.8 Ma traän nghòch ñaûo:

<u>*Ñònh nghóa:*</u>

i/ Ma traän vuoâng A^{-1} goïi laø ma traän nghòch ñaûo cuûa A $\in M_n(R)$
neáu $A.A^{-1}=A^{-1}A=I_n$.

ii/ A khaû ñaûo, k laø laø soá nguyeân döông, ta ñònh nghóa: $\left(A^k\right)^{-1} = \left(A^{-1}\right)^k$

<u>*Ví duï1:*</u> $A = \begin{bmatrix} -3 & 4 & 6 \\ 0 & 1 & 1 \\ 2 & -3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 0 & -3 \\ 2 & 1 & 3 \end{bmatrix}$

ta coù: AB=BA =I, neân $A^{-3} = \left(A^{-1}\right)^3 = (B)^3 = \begin{bmatrix} -3 & 4 & 4 \\ -20 & -29 & -34 \\ 12 & 22 & 21 \end{bmatrix}$

Hay $A^3 = \begin{bmatrix} -139 & 172 & 252 \\ -12 & 15 & 22 \\ 92 & -114 & -167 \end{bmatrix} \Rightarrow \left(A^3\right)^{-1} = \begin{bmatrix} -3 & 4 & -4 \\ -20 & -29 & -34 \\ 12 & 22 & 21 \end{bmatrix}$

<u>*Ñònh lyù:*</u> Neáu A, B $\in M_n$ (R) laø nhöõng ma traän khaû ñaûo thì:

i/ $\left(A^{-1}\right)^{-1} = A$, $(cA)^{-1} = \dfrac{1}{c}A^{-1}$, c ≠0 ($\in$R).

2i/ $(AB)^{-1} = B^{-1}A^{-1}$

3i/ $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

<u>*Ví duï2:*</u> $A \in M_{n\times n}$, vôùi $|A| \neq 0$, tìm ma traän nghòch ñaûo cuûa A. Bieát

$$A^2 + 3A - 2I_n = 0 \Leftrightarrow A\left(\frac{A+3I_n}{2}\right) = \left(\frac{A+3I_n}{2}\right)A = I_n \Rightarrow A^{-1} = \frac{A+3I_n}{2}$$

<u>*Ví duï3:*</u> Vôùi $P = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$, thì $P^{-1} = P^T = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$,

$P^n = \begin{bmatrix} \cos nx & -\sin nx \\ \sin nx & \cos nx \end{bmatrix}$ vaø $P^{-n} = \begin{bmatrix} \cos nx & \sin nx \\ -\sin nx & \cos nx \end{bmatrix}$. SV haõy kieåm tra.

<u>*Ví duï4:*</u> Tìm nghịch ñảo ma trận khối P, với A, D laø 2 ma traän vuoâng, $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, thì $P^{-1} = \left[\begin{array}{c|c} U & V \\ \hline Y & Z \end{array}\right]$,

vôùi ñk:A vaø $\left(D - CA^{-1}B\right)$ khaû ñaûo. Thì

$Z = \left(D - CA^{-1}B\right)^{-1}$, $Y = -ZCA^{-1}$, E ma traän ñôn vò.

$U = A^{-1} - A^{-1}BY$ $V = -A^{-1}BZ$, ta duøng PBĐSC ñeå chứng minh:

Vieát laïi: $\left[\begin{array}{cc|cc} A & B & E & 0 \\ \hline C & D & 0 & E \end{array}\right] \to \left[\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ \hline C & D & 0 & E \end{array}\right]$

$$\to\left[\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ \hline 0 & D-CA^{-1}B & -CA^{-1} & E \end{array}\right]\to\left[\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ \hline 0 & I & -ZCA^{-1} & ZE \end{array}\right]$$

$$\to\left[\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ \hline 0 & I & Y & Z \end{array}\right]\to\left[\begin{array}{cc|cc} I & A^{-1}B & A^{-1}-A^{-1}BY & -A^{-1}BZ \\ \hline 0 & I & Y & Z \end{array}\right]'$$

ÖÙng duïng: Tuy ma traän P coù kích thöôùc lôùn, nhöng neáu A, B, C, D coù kích thöôùc $\le$ 4 thì ta coù theå duøng maùy tính Vinacal-570MS ñeå tính nghòch ñaûo moät ma traän $p \in M_7$ nhö sau: $p=\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]$ trong ñoù:

a= $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 5 & 7 & 0 \\ 8 & -3 & 6 & 7 \\ 5 & -6 & -8 & 3 \end{bmatrix}$ b= $\begin{bmatrix} 9 & 0 & 3 \\ 7 & 0 & -3 \\ 0 & -9 & 7 \\ 9 & -4 & 8 \end{bmatrix}$ c= $\begin{bmatrix} 4 & 0 & 0 & 5 \\ 6 & 9 & -5 & -5 \\ 2 & 9 & -7 & 8 \end{bmatrix}$ d= $\begin{bmatrix} 6 & 7 & 8 \\ 4 & -5 & 0 \\ 4 & 0 & 0 \end{bmatrix}$

Vaäy p= $\begin{bmatrix} 2 & 3 & 4 & 5 & 9 & 0 & 3 \\ 3 & 5 & 7 & 0 & 7 & 0 & -3 \\ 8 & -3 & 6 & 7 & 0 & -9 & 7 \\ 5 & -6 & -8 & 3 & 9 & -4 & 8 \\ 4 & 0 & 0 & 5 & 6 & 7 & 8 \\ 6 & 9 & -5 & -5 & 4 & -5 & 0 \\ 2 & 9 & -7 & 8 & 4 & 0 & 0 \end{bmatrix}$ Ta tìm p^{-1} theo u, v, y, z nhö coâng thöùc ôû treân. Baám maùy Vinacal-570MS ñeå tìm: $p^{-1}=\left[\begin{array}{c|c} u & v \\ \hline y & z \end{array}\right]$,

ta coù caùc keát quaû sau:

Vaøo döõ lieäu	Pheùp tính trong maùy:
$a \to A$, $b \to B$, $c \to C$	$CA^{-1}B = Ans$
vaøo $d \to C$	$(C - Ans)^{-1} = \boxed{z} = Ans$
Vaøo laïi $c \to C$	$-Ans.C.A^{-1} = \boxed{y} = Ans$
	$A^{-1} - A^{-1}.B.Ans = \boxed{u}$
Goïi laïi	$CA^{-1}B = Ans$
vaøo laïi $d \to C$	$(C - Ans)^{-1} = \boxed{z} = Ans$
	$A^{-1}.B.Ans = \boxed{v}$

z= $\begin{bmatrix} -.02558996939 & -.007551767397 & -.01229225336 \\ .08911661385 & -.02802652057 & .01338790535 \\ .03204989561 & .08191518749 & -.07316809409 \end{bmatrix}$

$$y=\begin{bmatrix} .06077997755 & .03531813624 & -.03737055716 & .04874101644 \\ -.1019322193 & .06809725267 & -.02447797321 & -.003937335289 \\ .1943687315 & -.2026134746 & .006990222436 & -.06203466756 \end{bmatrix}$$

$$u=\begin{bmatrix} -.2613716872 & .1918852650 & .04989510095 & .04635154997 \\ .1055980551 & -.0871006920 & -.0029668475 & -.07339936831 \\ .05911016262 & .00068289824 & .02110192880 & -.04276413227 \\ -.0321234867 & .0329554301 & .02801339445 & .009197277785 \end{bmatrix}$$

$$v=\begin{bmatrix} -.07996159374 & .01494619043 & -.04299989484 \\ -.003733329593 & -.06928648141 & -.00509269008 \\ -.00238972048 & .00042655735 & .04113166336 \\ .00930440410 & .07080809797 & -.07867667123 \end{bmatrix}$$

Ví du 5: Cho $a := \begin{bmatrix} 2 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $b := \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $c := [-1 \quad 1 \quad 5]$, d=[3]

tìm ma tran $y = -zca^{-1}$ vôùi $z = \left(d - ca^{-1}b\right)^{-1}$

Ta xep chung lai thanh ma tran bac 4: $g := \begin{bmatrix} 2 & -2 & 1 & 3 \\ -1 & 0 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 5 & 3 \end{bmatrix}$, bam may

ñe co: $g^{-1}=\begin{bmatrix} \frac{3}{5} & \frac{7}{5} & \frac{4}{5} & \frac{-2}{5} \\ \frac{-4}{25} & \frac{-1}{25} & \frac{8}{25} & \frac{1}{25} \\ \frac{8}{25} & \frac{27}{25} & \frac{9}{25} & \frac{-2}{25} \\ \frac{-7}{25} & \frac{-33}{25} & \frac{-11}{25} & \frac{8}{25} \end{bmatrix}$ Vay y= $\begin{bmatrix} \frac{-7}{25} & \frac{-33}{25} & \frac{-11}{25} \end{bmatrix}$

Menh ñe: Neu A $\in M_n$ (R), cac phat bieu sau la töông ñöông:

i/ A la ma tran kha ñao.

ii/ A~I.

iii/ r(A)= n.

iv/ A la tích höu han cac ma tran sô cap.

2.9 Öng dung phep bien ñoi sô cap ñe tìm ma tran nghich ñao:

Vì ñac tính cua phep bien ñoi sô cap, co the bien ñoi ma tran vuong thanh ma tran ñôn vò neu ma tran vuong ño khong bò thoai hoa, lôi dung ñieu nay va do tính chat ma tran nghich ñao $A.A^{-1}=I_n= A^{-1}.A$, ta co the söû dung cac phep bien doi ngöôc lai ñe bien ma tran ñôn vò thanh lai ma tran ban ñau va ñao lai nhö sô ño sau.

$$\left(A \middle| I_n\right) \rightarrow \mathbf{L}\ \mathbf{L} \rightarrow \left(I_n \middle| A^{-1}\right)$$

Ví du 1: Tìm ma tran nghich ñao cua

$$A=\begin{bmatrix}1&2&3\\0&-1&2\\3&2&5\end{bmatrix}\to\left(A\middle|I_n\right)=\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&-1&2&0&1&0\\3&2&5&0&0&1\end{array}\right]\xrightarrow{-3d_1+d_3}$$

$$\to\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&-1&2&0&1&0\\0&-4&-4&-3&0&1\end{array}\right]\to$$ chu y: $S_1=\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}$

$$\xrightarrow{d_3/4}\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&-1&2&0&1&0\\0&-1&-1&-\frac{3}{4}&0&\frac{1}{4}\end{array}\right]\xrightarrow{d_2\leftrightarrow(-d_3)}\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&1&\frac{3}{4}&0&-\frac{1}{4}\\0&-1&2&0&1&0\end{array}\right]$$

ta co: $S_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{4}\end{bmatrix}$, $S_3=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$,

$$\xrightarrow{d_2+d_3}\left[\begin{array}{ccc|ccc}1&0&1&-\frac{1}{2}&0&\frac{1}{2}\\0&1&1&\frac{3}{4}&0&-\frac{1}{4}\\0&0&3&\frac{3}{4}&1&-\frac{1}{4}\end{array}\right]\xrightarrow{d_3/3}\left[\begin{array}{ccc|ccc}1&0&0&-\frac{3}{4}&-\frac{1}{3}&\frac{7}{12}\\0&1&0&\frac{1}{2}&-\frac{1}{3}&-\frac{1}{6}\\0&0&1&\frac{1}{4}&\frac{1}{3}&-\frac{1}{12}\end{array}\right]$$

ta co:

$$S_4=\begin{bmatrix}1&-2&0\\0&1&0\\0&1&1\end{bmatrix},S_5=\begin{bmatrix}1&0&-\frac{1}{3}\\0&1&-\frac{1}{3}\\0&0&\frac{1}{3}\end{bmatrix},\Rightarrow A^{-1}=\begin{bmatrix}-\frac{3}{4}&-\frac{1}{3}&\frac{7}{12}\\\frac{1}{2}&-\frac{1}{3}&-\frac{1}{6}\\\frac{1}{4}&\frac{1}{3}&-\frac{1}{12}\end{bmatrix}$$

Ô ñay ta co the kiem chöng A la tích höu han cac ma tran sô cap nhö sau: Theo menh ñe ña hoc ô tren, can cö vao phep bien ñoi sô cap ô tren ta co lan lööt cac ma tran sô cap S_1, S_2, S_3, S_4, S_5, sao cho: I = S_5. S_4. S_3. S_2. S_1. A <=>

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\underbrace{\begin{bmatrix}1&0&-\frac{1}{3}\\0&1&-\frac{1}{3}\\0&0&\frac{1}{3}\end{bmatrix}}_{S_5}\underbrace{\begin{bmatrix}1&-2&0\\0&1&0\\0&1&1\end{bmatrix}}_{S_4}\underbrace{\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}}_{S_3}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{4}\end{bmatrix}}_{S_2}\times$$

$$\times\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}}_{S_1}\underbrace{\begin{bmatrix}1&2&3\\0&-1&2\\3&2&5\end{bmatrix}}_{A}$$. Vay $A=S_1^{-1}S_2^{-1}S_3^{-1}S_4^{-1}S_5^{-1}$, kiem chöng! ¢

Ta cung co the dung PBÑSC tren <u>dong</u> ñe tính $A^{-1}B$ nhö sau:

Ví duï 1: Ñeå tính ma traän tích cuûa $A^{-1}B$, ta thieát laäp $\left[A|B\right]$, sau ñoù duøng PBÑSC treân doøng ñeå ñöa $\left[A|B\right]\to\ldots..\to\left[I\,\middle|\,A^{-1}B\right]$.

Đặc biệt B= ma trận cột, thì giải Ax=B tương ñương với duøng PBĐSC ñeå bieán $\left[A|B\right]\to\ldots\to\left[I\,\middle|\,A^{-1}B=x\right]$. Xem

Ma traän $A=\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}, B=\begin{bmatrix}1&0&0\\2&1&0\\3&0&1\end{bmatrix}$, tính $A^{-1}B$. Ta thieát laäp ma traän:

$$\left[A|B\right]=\left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&1&2&1&0\\1&0&1&3&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&1&2&1&0\\0&1&-1&-2&0&-1\end{array}\right]\to$$

$$\to\left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&1&2&1&0\\0&0&2&4&1&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&2&0&0&1&-1\\0&0&2&4&1&1\end{array}\right]\to$$

$$\to\left[\begin{array}{ccc|ccc}2&0&0&2&-1&1\\0&2&0&0&1&-1\\0&0&2&4&1&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&1&\frac{-1}{2}&\frac{1}{2}\\0&1&0&0&\frac{1}{2}&\frac{-1}{2}\\0&0&1&2&\frac{1}{2}&\frac{1}{2}\end{array}\right]=\left[I\,\middle|\,A^{-1}B\right]$$

Vaäy $A^{-1}B=\left[\begin{array}{c|cc}1&\frac{-1}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{-1}{2}\\2&\frac{1}{2}&\frac{1}{2}\end{array}\right]$. Đặc biệt ở ví dụ naøy, baây giờ giải: Ax=B với

$A=\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}, B=\begin{bmatrix}1\\2\\3\end{bmatrix}$ thì $A^{-1}B=x=\begin{bmatrix}1\\0\\2\end{bmatrix}$ chính laø coät moät cuûa keát quaû $A^{-1}B$ ôû treân ¢

BAØI TAÄP

1/ Cho $\mathbf{A}=\begin{bmatrix}1&-2&0&10\\3&6&1&8\end{bmatrix}$ $\mathbf{B}=\begin{bmatrix}-1&-2&6&5\\3&0&8&9\end{bmatrix}$

a/ Tính 2A-3B.

b/ Tính X sao cho A+X=B.

2/ Cho $\mathbf{A}=\begin{bmatrix}1&3&5&-7\\0&9&-8&2\end{bmatrix}$ vaø $B=\begin{bmatrix}-2&6&3&5\\0&8&2&9\end{bmatrix}$

a/ Tính X sao cho 2A-3X=B.

b/ Tính B^T, A B^T, BA^T.

3/ Cho 2 ma traän $\mathbf{A} = \begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 4 & 5 \end{bmatrix}$, vaø $\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 9 & 0 \end{bmatrix}$

a/ Tính A.B vaø det(A.B).

b/ Tính B.A vaø det(B.A)

4/ Tính ma traän nghòch ñaûo caùc ma traän sau:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -6 & 0 \\ 7 & -2 & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -7 & 2 \\ 0 & 1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 1/3 & 1/2 \\ 2/5 & 1/4 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 1 & -5 & \frac{-7}{3} & \frac{-2}{3} \\ 0 & 1 & \frac{-1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 1 & -5 & 4 & -1 \\ 0 & 1 & \frac{1}{3} & \frac{-1}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5/ Tính caùc ma traän X sao cho:

a/ $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}.X = \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}.X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, c/ $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.X = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$

6/ Tìm X sao cho AX=B nhö sau:

a/ $\begin{bmatrix} 2 & 3 & 11 & 5 \\ 1 & 1 & 5 & 2 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 5 & 4 \end{bmatrix}.X = \begin{bmatrix} 2 \\ 1 \\ -3 \\ -3 \end{bmatrix}$ b/ $\begin{bmatrix} 2 & -3 & 5 & 7 \\ 4 & -6 & 2 & 3 \\ 2 & -3 & -11 & -15 \end{bmatrix}.X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

c/ $\begin{bmatrix} 2 & 5 & -8 \\ 4 & 3 & -9 \\ 2 & 3 & -5 \\ 1 & 8 & -7 \end{bmatrix}.X = \begin{bmatrix} 8 \\ 9 \\ 7 \\ 12 \end{bmatrix}$ d/ $\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 3 & 2 & 5 \\ 9 & 1 & 4 & -5 \\ 2 & 2 & 3 & 4 \\ 7 & 1 & 6 & -1 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 5 \\ 7 \end{bmatrix}$

7/ Giaûi vaø bieän luaän theo m caùc heä phöông trình sau:

a/ $\begin{cases} x+y+z=2 \\ 2x+y-2z=2 \\ mx+y+4z=2 \end{cases}$ b/ $\begin{cases} mx+y+z=0 \\ x+my+z=0 \\ x+my+z=m+1 \end{cases}$ c/ $\begin{cases} mx+y+z=1 \\ x+my+z=m \\ x+my+z=m^2 \end{cases}$

d/ $\begin{cases} x + y + (1-m)z = m+2 \\ (1-m)x - y + 2z = 0 \\ 2x - my + 3z = m+2 \end{cases}$ e/ $\begin{cases} x+y+z+t=3 \\ x+y = 4 \\ x+y+mz+t=2 \end{cases}$

f/ $\begin{cases} mx + y + z = m \\ 2x+(m+1)y+(m+1)z=(m+1) \\ x + y + mz = 1 \end{cases}$

8/ Ngöôøi ta ñònh nghóa haøm muõ ma traän (gioáng nhö haøm muõ thöïc) nhö sau: cho $A = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$, a,b $\in$ R,

$$e^{A}=I_2+\frac{A}{1!}+\frac{A^2}{2!}+\frac{A^3}{3!}+\mathrm{L}+\frac{A^n}{n!}+\mathrm{L}\ .$$

Tính. e^A. Trong ñoù $I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ laø ma traän vuoâng ñôn vò.

9/ Giaû söû A laø ma traän vuoâng vaø coù k ≥ sao cho $A^k=0$.

a/ Cm B=I-A khaû ñaûo vaø tính B^{-1} theo I vaø A.

b/ Cm C=I+A khaû ñaûo vaø tính C^{-1} theo I vaø A.

9/ Cho $\mathbf{E}=\begin{bmatrix}1&2&4\\5&1&2\\3&-1&1\end{bmatrix}$. Tìm S_1, S_2, S_3: khaû ñaûo $\in M_3(R)$: $E=S_1\,S_2\,S_3$.

10/ Cho $f(x)=x^3-7x+5$, $g(x)=2x^3+3x-4$, $\mathbf{A}=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$

Tính f(A), g(A), f(A).g(A), ví duï $g(A)=2A^3+3A-4I$, trong ñoù I laø ma traän ñôn vò.

CHÖÔNG 2
ÑÒNH THÖÙC
BAØI 1: ÑÒNH THÖÙC

1.1 Ñònh nghóa:

<u>*Hoaùn vò:*</u> Ta ñaõ bieát ôû phoå thoâng, moät hoaùn vò cuûa n soá töï nhieân 1,2,3,..., n goïi laø moät hoaùn vò baäc n (laø song aùnh: $n\ \mathrm{a}\ f(n)$).

<u>*Nghòch theá:*</u> Cho hoaùn vò $(i_1,i_2,i_3,\mathrm{L},i_n)$, caëp soá (i_j,i_k) trong hoaùn vò ñoù taïo thaønh moät nghòch theá neáu $i_j>i_k$ vaø i_j ñöùng tröôùc (beân traùi) i_k trong hoaùn vò ñoù.

<u>*Ví duï1:*</u> Trong hoaùn vò 3241 ta coù 4 nghòch theá (3,2), (3,1), (2,1), (4,1),

Moät hoaùn vò ñöôïc goïi laø chaün (leû), neáu soá nghòch theá trong hoaùn vò ñoù laø chaün (leû)

- Ñònh thöùc caáp 2 laø moät soá xaùc ñònh nhö sau:

$$\textbf{(1)}\quad \Delta_2=\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}=a_1b_2-a_2b_1$$

Soá haïng $a_{11}a_{22}$ coù daáu döông do hoaùn vò (1,2) chaün.

Soá haïng $a_{12}a_{21}$ coù daáu aâm do hoaùn vò (2,1) leû.

<u>*Ví du 2*</u>: $\begin{vmatrix}4&-5\\-1&-2\end{vmatrix}=4(-2)-(-5)(-1)=-13$.

- Ñònh thöùc caáp 3 laø moät soá xaùc ñònh nhö sau:

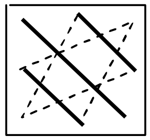
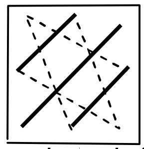
$$\textbf{(2)}\quad \Delta_3=\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}=a_1.\begin{vmatrix}b_2&c_2\\b_3&c_3\end{vmatrix}-b_1.\begin{vmatrix}a_2&c_2\\a_3&c_3\end{vmatrix}+c_1.\begin{vmatrix}a_2&b_2\\a_3&b_3\end{vmatrix}$$

$$=a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_3b_1c_2+a_2b_3c_1-a_3b_2c_1$$

ví duï nhö soá haïng $a_1b_3c_2$ coù daáu aâm do hoaùn vò (1,3,2) leû.

$-a_3b_2c_1$ coù 3 nghòch theá neân mang daáu aâm.

Ñeå deã nhôù ta coù qui taéc Sarius sau:

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$  $-$ 

Qui tac nay con ñöôc goi la 3 ñöông cheo döông trö 3 ñöông cheo am.

$$\textbf{(3)} \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$= a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

• Ñònh thöc cap n: Xet ñònh thöc cap n la bang sau ñay:

$$D(A) = |A| = \det(A) = \begin{vmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{vmatrix}$$

Ñay la mot ham nhieu bien $f : R^n \times R^n \to R$

• Ñònh thöc con bu cua phan tö a_{ij}, ky hieu la M_{ij}, la ñònh thöc bac (n-1) nhan ñöôc tö D(A) bang cach xoa ñi dong thö i va cot thö j.

• Phan bu ñai so cua phan tö a_{ij}, ky hieu la A_{ij}, ñöôc ñònh nghóa nhö sau:

$\overline{A} = \left[A_{ij}\right]$ trong ño $A_{ij} = (-1)^{i+j} M_{ij}$

1.2 Ñònh ly:

Cong thöc trien khai Laplace, khai trien hang thö i, nhö sau:

$$\textbf{(*)}\ D(A) = \sum_{j=1}^{n} a_{ij}A_{ij} = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + \cdots + a_{in}A_{in}$$

khai trien cot thö j, nhö sau:

$$\textbf{(*)}\ D(A) = \sum_{i=1}^{n} a_{ij}A_{ij} = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + \cdots + a_{nj}A_{nj}$$

Ô dang (3) ô tren, khai trien cot, ta co

$$\textbf{(**)} \quad D(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3,$$

Ví du2:

$$\Delta = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 3 & 1 & 0 & 2 \\ 3 & 1 & 0 & 2 \end{vmatrix}.$$ Ap dung cong thöc laplace, khai trien theo cot thö ba ta co:

$$\Delta = (-1)^{2+3}.1.\begin{vmatrix} 1 & -2 & 5 \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix} =$$

$$= -\left(\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} \right) = 0 .$$

1.3 Cac tính chat cua ñònh thöùc:

Ñe minh hoa cho cac tính chat cua ñònh thöùc, ta minh hoa cho ñònh thöùc bac 3, va ñònh thöùc bac n van co hieu löïc töông töï. Cac tính chat phat bieu cho hang hay cot ñeu co vai tro nhö nhau. Ñe kiem chöùng cac tính chat sau ta chæ can döïa vao ñònh nghóa.

<u>Tính chat 1:</u> Neu ñoi hang va cot cho nhau thì gia trò ñònh thöùc khong ñoi. $|A| = |A^T|$. (A_i la phan bu ñai so cua a_i)

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3$$ do khai trien theo hang 1

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3$$ do khai trien theo cot 1

<u>Ví du1</u>: $\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$ = 5 -6 = -1

<u>Ví du2</u>: $\begin{vmatrix} 1 & 0 & -2 \\ -2 & 1 & 1 \\ 3 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ -2 & 1 & 3 \end{vmatrix} =$

= [1.1.3+0+(-2) (-1) (-2)]-[3.(-2)1+1(-1)1]=6

nghóa la: det(A) =det(A^T)

<u>Tính chat 2:</u> Neu ñoi cho hai hang (cot) cho nhau thì gia trò ñònh thöùc ñoi dau. (B_i la phan bu ñai so cua b_i)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = b_1B_1 + b_2B_2 + b_3B_3$$ khai trien cot 2, co dau (-, +, -)

$$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = b_1B_1^/ + b_2B_2^/ + b_3B_3^/$$ khai trien cot 1, co dau (+, -, +)

nen $B_i^/ = -B_i$

<u>Ví du3</u>: $\begin{vmatrix} 1 & 0 & -2 \\ -2 & 1 & 1 \\ 3 & -1 & 3 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & -2 \\ 1 & -2 & 1 \\ -1 & 3 & 3 \end{vmatrix}$

(ñoi cho cot 1 va cot 2 thì dau ñoi).

<u>He qua:</u> Co ít nhat hai hang (cot) gioáng nhau thì ñònh thöùc bang 0.

Xem lai ví du 2 muc 1.2.

<u>Tính chat 3:</u> Thöøa so chung cua mot hang (cot) thì ñöôïc mang ra ngoai.

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = kb_1B_1 + kb_2B_2 + kb_3B_3 =$$

$$= k\left(b_1B_1 + b_2B_2 + b_3B_3\right) = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

<u>*Ví du5*</u>: $\begin{vmatrix} 3\times 1 & 0 & -2 \\ 3\times(-2) & 1 & 1 \\ 3\times 3 & -1 & 3 \end{vmatrix} = 3.\begin{vmatrix} 1 & 0 & -2 \\ -2 & 1 & 1 \\ 3 & -1 & 3 \end{vmatrix}$

<u>*Ví du6*</u>: $B = a A \Rightarrow |B| = a^n |A|$

$$\begin{vmatrix} a a_{11} & a a_{12} & \cdots & a a_{1n} \\ a a_{21} & a a_{22} & \cdots & a a_{2n} \\ \vdots & & \ddots & \vdots \\ a a_{n1} & a a_{n2} & \cdots & a a_{nn} \end{vmatrix} = a^n \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

<u>*Tính chat 4:*</u> Neu co mot hang (hay cot) ma moi so hang la tong cua 2 so hang thanh phan thì ñònh thöùc co the tach ra thanh tong hai ñònh thöùc.

$$\begin{vmatrix} a_1 & b_1+d_1 & c_1 \\ a_2 & b_2+d_2 & c_2 \\ a_3 & b_3+d_3 & c_3 \end{vmatrix} = \left(b_1B_1 + b_2B_2 + b_3B_3\right) + \left(d_1B_1 + d_2B_2 + d_3B_3\right)$$

<u>*Ví du7*</u>: $\begin{vmatrix} 3+1 & 0 & -2 \\ 5+(-2) & 1 & 1 \\ 4+3 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 0 & -2 \\ 5 & 1 & 1 \\ 4 & -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -2 \\ -2 & 1 & 1 \\ 3 & -1 & 3 \end{vmatrix}$

<u>Ví du tong quat</u>: Gia sö ñònh thöùc bac 3, cot mot la tong 2 so hang A, va B, cot hai la tong 2 so hang C, va D, cot ba la tong 2 so hang E, va F.

$$|A+B \quad C+D \quad E+F| = |A \quad C \quad E| + |A \quad C \quad F|$$

(co 2×2×2 =2^3=8 thanh phan tach theo cot)

$$+ |A \quad D \quad E| + |A \quad D \quad F| + |B \quad C \quad E| + |B \quad C \quad F| + |B \quad D \quad E| + |B \quad D \quad F|$$

Va $|A+B+C \quad D+E+F \quad G+H|$ (co 3×3×2=18 thanh phan)

<u>*Tính chat 5:*</u> He qua cua cac tính chat tren la ñònh thöùc se khong ñoi neu ta cong vao mot hang (hay cot) vôi boi so mot hang khac (sau khi ña nhan vôi mot hang so).

Ta thöông dung tính chat 5 ñe tính gia trò ñònh thöùc.

$$\begin{vmatrix} a_1+kc_1 & b_1 & c_1 \\ a_2+kc_2 & b_2 & c_2 \\ a_3+kc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0$$

<u>*Ví dụ 10*</u>: Lấy lại ví dụ trước, tính ñònh thức bậc 5:

$$\begin{vmatrix} 1 & \boxed{2} & 3 & 1 & -1 \\ 2 & 1 & 2 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 3 & 3 \\ 2 & 0 & 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -3 & 0 & -1 & 3 & -3 \\ 2 & 1 & 2 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 3 & 3 \\ 2 & 0 & 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -3 & -1 & 3 & -3 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & \boxed{3} & 3 \\ 2 & 1 & 0 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & -1 & 3 & -3 \\ -1 & 1 & 0 & 1 \\ 4 & 0 & 0 & 6 \\ 2 & 1 & 0 & 2 \end{vmatrix} = 3\begin{vmatrix} -1 & 1 & 1 \\ 4 & 0 & 6 \\ 2 & \boxed{1} & 2 \end{vmatrix} = 3\begin{vmatrix} -1 & 1 & 1 \\ 4 & 0 & 6 \\ 3 & 0 & 1 \end{vmatrix} = -3\begin{vmatrix} 4 & 6 \\ 3 & 1 \end{vmatrix} = -3 \times -14 = 42$$

Tính chat 6: Tích vo höông cua mot hang vôi phan bu ñai so cua phan tö tren mot hang khac thì bang 0.

Trong khi theo ñònh ly Laplace, tích vo höông cua mot hang vôi phan bu ñai so cua chính hang ño, thì bang det(A) nhö (**) va (***) ô tren (khai trien Laplace).

That vay coi ñònh thöc co cot 1 va cot 3 giong nhau, khai trien theo cot 1 va chu y cac ky hieu ña ñöôc ñònh nghóa ta co:

$$\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} = c_1A_1 + c_2A_2 + c_3A_3 = 0$$

Töông tö: $a_1A_1 + b_1B_1 + c_1C_1 = \det(A)$, $a_1A_2 + b_1B_2 + c_1C_2 = 0$ .....

Ví du: nhan vo höông cua cot 1 vôi phan bu ñai so cua cot 2 cua ma tran sau:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} . \begin{bmatrix} -3 \\ -7 \\ -2 \end{bmatrix} = -3 + 7 - 4 = 0$$

CHÖÔNG 3
KHONG GIAN VECTÔ (KGVT)
BAI 1: KGVT

1.1 Ñònh nghóa: Khong gian vectô (KGVT) la cap (V, R), R la tröông so thöc, V la tap cac phan tö vectô, trang bò 2 phep toan cong va nhan nhö sau:

$(V,V) \rightarrow V$ $\quad\quad$ $(R,V) \rightarrow V$

$(x_1, x_2) \mapsto x_1 + x_2$ $\quad\quad$ $(\lambda, x) \mapsto \lambda.x$.

Va hai phep toan nay phai thoa tam tính chat sau:

1/ $x_1 + x_2 = x_2 + x_1$.
2/ $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.
3/ $\exists!\, 0 \in V$: $0 + x = x + 0 = x$.
4/ $\exists\, (-x) \in V$: $-x + x = x - x = 0$.
5/ $(\lambda_1 . \lambda_2)\, x = \lambda_1 . (\lambda_2 x)$.
6/ $\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2$.
7/ $(\lambda_1 + \lambda_2) x = \lambda_1 x + \lambda_2 x$.
8/ $1.x = x$.

Mot vai tính chat: co the suy ra tö ñònh nghóa:

- $\forall v \in V, \quad -v = (-1).v$

- $\forall v \in V, \forall a \in R, \quad (-a\ v) = (-a)v = a(-v)$
- $\forall a \in R, \quad a.0 = 0$
- Neu $\alpha v=0$ thì $\alpha=0$ hay $v=0$.
- $\begin{cases} a\ v = b\ v \\ v \neq 0 \end{cases} \Rightarrow a = b$, $\begin{cases} a\ v = a\ u \\ a \neq 0 \end{cases} \Rightarrow v = u$

Ví du1: Tap cac vectô hình hoc Euclid thong thöông tao thanh KGVT

Ví du2: Tap cac vectô $\{\vec{x} : \vec{x} = k\vec{a}, k \in R\}$, ($\vec{a}$ vectô co ñònh) tao thanh KGVT.

Ví du3: Tröông so thöc R la KGVT tren chính no.

Ñònh nghóa: Tap W, $\phi \neq W \subset V$ ñöôc goi la khong gian con cua KGVT V neu cac phep toan cong va nhan cua V han che len W.

Ñònh ly: Tap W, $\phi \neq W \subset V$ la khong gian con cua KGVT V neu va chæ neu $\forall x,y \in W$, $\forall \alpha, \beta \in R$, ta co $\alpha x + \beta y \in W$.

Ví du: trong R^n, W=$\left\{x = (x_1, x_2, \ldots x_n)^T : Ax = 0\right\}$ la mot KGVT con cua R^n, trong ño $A \in M_{m,n}$. De dang kiem chöng. Thöc ra ñay la mot ñònh ly.

1.3.1 To hôp tuyen tính (THTT):

Xet cac vectô $A_1, A_2, \ldots, A_m \in V$ va cac so thöc $k_1, k_2, \ldots, k_m$. Mot to hôp tuyen tính cua cac vectô $A_1, A_2, \ldots, A_m$ la vectô:

$$A = k_1 A_1 + k_2 A_2 + \ldots + k_m A_m = \sum_{i=1}^{i=m} k_i A_i$$

Ví du : Trong R^3: $A_1 = (1, -1, 0)$, $A_2 = (2, 1, -2)$, thì mot THTT cua chung la: $A = 2A_1 - A_2 = (2, -2, 0) - (2, 1, -2) = (0, -3, 2)$.

1.3.2 Phu thuoc tuyen tính(PTTT), ñoc lap tuyen tính (ÑLTT):

Ñònh nghóa: Cac vectô $V_1, V_2, \ldots, V_m$ goi la phu thuoc tuyen tính neu ton tai cac so thöc $k_1, k_2, \ldots, k_m$. khong ñong thôi bang 0, sao cho:

$$k_1 V_1 + k_2 V_2 + \ldots + k_m V_m = 0$$

Neu cac vectô $V_1, V_2, \ldots, V_m$ khong phu thuoc tuyen tính, ta noi chung ñoc lap tuyen tính.

Noi cach khac: $V_1, V_2, \ldots, V_m$ goi la PTTT <==>
$k_1 V_1 + k_2 V_2 + \ldots + k_m V_m = 0$ va $\exists i \in \overline{1,m} \quad k_i \neq 0$

Ñònh ly: Cac vectô $V_1, V_2, \ldots, V_m$ la ñoc lap tuyen tính (ÑLTT)
<==> $k_1 V_1 + k_2 V_2 + \ldots + k_m V_m = 0$ ==> $k_1 = k_2 = \ldots\ldots k_m = 0$

ñònh ly ñöôc suy ra hien nhien, tö ñònh nghóa cua PTTT.

Ví du1: Cac vectô u = (1, -1, 0), v = (1, 3, -1), w = (5, 3, -2) la phu thuoc tuyen tính vì:
3u + 2v – w = 3 (1, -1, 0) + 2 (1,3-1) – (5,3, -2) = (0,0,0)

Ví du2: Cac vectô u = (1, -2, 1), v = (2, 1, -1), w = (7, -4, 1) la phu thuoc tuyen tính vì:

au + bv +c w = (0,0,0) <==> $\begin{cases} a+2b+7c=0 \\ -2a+b-4c=0 \\ a-b+c=0 \end{cases}$, xeùt

A=$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix} \rightarrow \mathbf{L} \rightarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, r(A)=2 < n=3, heä thuaàn nhaát coù nghieäm khoâng taàm thöôøng (nghieäm a, b, c khoâng ñoàng thôøi baèng khoâng), vaäy heä PTTT. Do ñoù ta coù ñònh lyù:

Ñònh lyù: Ñieàu kieän caàn vaø ñuû ñeå caùc vectô $V_1, V_2, \ldots, V_m \in R^n$ laø PTTT laø rank $(V_1, V_2, \ldots, V_m) < m$.

Chuù yù: Ta luoân coù rank $(V_1, V_2, \ldots, V_m) \le \min(m,n)$

Ví duï 3: Trong R^5, u = (1, 2, 3, 4,5), v = (-1,2,-3,4,5), w = (1,-2, 3, 4,-5). Ta coù m=3, n=5.

A=$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -2 \\ 3 & -3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & -5 \end{bmatrix} \rightarrow \mathbf{L} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, neân rank(A)=3=m, ÑLTT

- Taäp hôïp Vect$(A_1, A_2, \ldots, A_m)$ = $\langle A_1, A_2, \ldots, A_m \rangle$ = $=\left\{ \sum_{i=1}^{i=m} k_i A_i, k_i \in R \right\}$ laø moät KGVT con, sinh bôûi höõu haïn caùc vectô $A_1, A_2, \ldots, A_m$. Ñöôïc goïi laø toå hôïp tuyeán tính cuûa heä $A_1, A_2, \ldots, A_m$.

Ví duï 10: W laø KGVT con cuûa R^4 goàm caùc vectô (x_1, x_2, x_3, x_4) thoûa:

$\begin{cases} x_1+2x_2+3x_3+x_4=0 \\ 2x_1+4x_2+7x_3+x_4=0 \\ 4x_1+8x_2+13x_3+3x_4=0 \end{cases}$, $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 1 \\ 4 & 8 & 13 & 3 \end{bmatrix} \rightarrow \mathbf{L} \rightarrow$

$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -2x_2 - 4x_4 \\ x_3 = x_4 \end{cases}$,

$(x_1, x_2, x_3, x_4) = x_2(-2,1,0,0) + x_4(-4,0,1,1)$.

Vaäy W=$\langle(-2,1,0,0),(-4,0,1,1)\rangle$.

Ví duï 11: Tìm cô sôû vaø soá chieàu cuûa khoâng gian nghieäm:

$\begin{cases} 3x_1+4x_2+x_3+2x_4+3x_5=0 \\ 5x_1+7x_2+x_3+3x_4+4x_5=0 \\ 4x_1+5x_2+2x_3+x_4+5x_5=0 \\ 7x_1+10x_2+x_3+6x_4+5x_5=0 \end{cases}$, duøng PBÑSC: $\begin{bmatrix} 3 & 4 & 1 & 2 & 3 \\ 5 & 7 & 1 & 3 & 4 \\ 4 & 5 & 2 & 1 & 5 \\ 7 & 10 & 1 & 6 & 5 \end{bmatrix}$

$$\begin{bmatrix}3&4&1&2&3\\5&7&1&3&4\\4&5&2&1&5\\7&10&1&6&5\end{bmatrix}\to\begin{bmatrix}3&4&1&2&3\\0&1&-2&-1&-3\\0&0&0&-2&0\\0&0&0&0&0\end{bmatrix}$$

$x_4 = 0$, $x_2 = 2x_3 + 3x_5$, $x_1 = \frac{1}{3}(-4x_2 - x_3 - 3x_5) = -3x_3 - 5x_5$

$\langle x_1,x_2,x_3,x_4,x_5\rangle = \langle x_3(-3,2,1,0,0)+x_5(-5,3,0,0,1)\rangle$,

E = $\langle(-3,2,1,0,0),(-5,3,0,0,1)\rangle$, chöng minh

E la mot KGVT con (mot sieu phang) cua R^5.

Ví du12: Tröng ra cau truc khong gian nghiem cua he phöông trình:

$$(1)\begin{cases}2x_1-1x_2+1x_3+2x_4+3x_5=2\\6x_1-3x_2+2x_3+4x_4+5x_5=3\\6x_1-3x_2+4x_3+8x_4+13x_5=9\\4x_1-2x_2+1x_3+1x_4+2x_5=1\end{cases}$$ <==>Ax=b

$$\left[\begin{array}{ccccc|c}2&-1&1&2&3&2\\6&-3&2&4&5&3\\6&-3&4&8&13&9\\4&-2&1&1&2&1\end{array}\right]\to \mathbf{L}\to\left[\begin{array}{ccccc|c}2&-1&1&2&3&2\\0&0&-1&-2&-4&-3\\0&0&0&-1&0&0\\0&0&0&0&0&0\end{array}\right]$$

$$\to\left[\begin{array}{ccccc|c}2&-1&0&0&-1&-1\\0&0&-1&-2&-4&-3\\0&0&0&-1&0&0\\0&0&0&0&0&0\end{array}\right]$$ => $x_4 = 0$, $\begin{cases}x_3=3-4x_5\\x_2=2x_1-x_5+1\end{cases}$

$\langle x_1,x_2,x_3,x_4,x_5\rangle=\langle x_1,2x_1-x_5+1,3-4x_5,0,x_5\rangle=$

$=\underbrace{(0,1,3,0,0)}_{\vec{a}}+x_1\underbrace{(1,2,0,0,0)}_{\vec{b}}+x_5\underbrace{(0,-1,-4,0,1)}_{\vec{c}}$. Khong gian nghiem la 1 sieu phang tao bôi 2 vectô $\vec{b}$ va $\vec{c}$ trong R^5 sau khi ña tình tien ñi bôi mot vectô $\vec{a}$. Nhöng ñay co phai la mot KGVT ?

Thöc ra ta se thay ro hôn (ví du phan sau): ô ñay $\vec{a}$ chính la mot nghiem rieng cua **(1)**, con $\vec{b}$, $\vec{c}$ chính la mot cô sô cua kerA (Ax=0). Va tat nhien cach bieu dien nay khong duy nhat. Ô ví du tren co the bieu dien:

$\langle x_1,x_2,x_3,x_4,x_5\rangle=(0,2,7,0,-1)+a(1,2,0,0,0)+b(0,-1,-4,0,1)$

Menh ñe: Gia sö $W_1=\langle S_1\rangle$, $W_2=\langle S_2\rangle \Rightarrow W_1+W_2=\langle S_1\cup S_2\rangle$.

Ví du13: Trong R^4 cho $v_1=(1,1,0,1), v_2=(2,3,1,2), v_3=(3,4,1,3)$,

Goi $W_1=\langle v_1,v_2\rangle$, $W_2=\langle v_2,v_3\rangle$. Tìm ñieu kien ñe

$v=(a,b,c,d)\in W=W_1+W_2$. Nghia la tìm W.

Theo menh ñe tren ta co $W=W_1+W_2=\langle S_1\cup S_2\rangle=\langle v_1,v_2,v_3\rangle$, nghia la v la to hôp tuyen tính cua v_1, v_2, v_3, nghóa la ta co:

toàn taïi (a_1,a_2,a_3): $\begin{cases} a_1+2a_2+3a_3=a \\ a_1+3a_2+4a_3=b \\ \quad\; a_2+a_3=\; c \\ a_1+2a_2+3a_3=d \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 1 & 3 & 4 & b \\ 0 & 1 & 1 & c \\ 1 & 2 & 3 & d \end{array}\right]$

$$\rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 1 & c \\ 0 & 0 & 0 & d-a \\ 0 & 0 & 0 & b-a-c \end{array}\right]$$

Heä phöông trình naøy coù nghieäm neáu vaø chæ neáu $\begin{cases} b=a+c \\ d=a \end{cases}$, suy ra

$$W=W_1+W_2=\left\{(a,b,c,d)\in R^4 : b=a+c, d=a\right\}=\left\langle \underbrace{(1,1,0,1)}_{v_1},\underbrace{(0,1,1,0)}_{v_4} \right\rangle.$$

Phaân tích theâm: $V_1=[v_1 \mid v_2 \mid v_3]=\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{array}\right] \rightarrow \ldots \rightarrow \left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

==> rank(V_1)=2, vaø $V_2=[v_1 \mid v_2]$, rank(V_2)=2, vaø laïi ta coù: $v_1+v_2=v_3$

$V_3=[v_1 \mid v_3]$, rank(V_3)=2, $V_4=[v_2 \mid v_3]$, rank(V_4)=2, neân

$W=W_1+W_2=\langle v_1,v_2\rangle=\langle v_1,v_3\rangle=\langle v_2,v_3\rangle=\langle v_1,v_2,v_3\rangle$.

Sinh vieân haõy chöùng minh cuï theå töøng daáu baèng cuûa ñaúng thöùc.

BAØI 2: CÔ SÔÛ VAØ CHIEÀU KHOÂNG GIAN

2.1 Cô sôû vaø chieàu kgvt:

Moät KGVT toång quaùt (V, R) coù heä höõu haïn vaø saép thöù töï (ñöôïc saép thöù töï). $P=\{x_1,x_2,x_3,\cdots,x_p\}\subset V$ ñöôïc goïi laø cô sôû cuûa V neáu vaø chæ neáu heä P naøy ÑLTT vaø ∀x∈V ñeàu ñöôïc sinh bôûi P nghóa laø toàn taïi duy nhaát caùc $\alpha_i \in$ R, ñeå ta coù:

$$x=a_1x_1+a_2x_2+a_3x_3+\cdots+a_px_p=\sum_{i=1}^{P} a_ix_i,$$

- Caùc α_i goïi laø toaï ñoä cuûa vectô x trong cô sôû P, kí hieäu:

  $[x]_P=(a_1,a_2,...,a_p)^T=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$ vaø goïi $[x]_P$ laø toïa ñoä cuûa x trong cô sôû P.
- Noùi khaùc ñi KGVT (V, R) ñöôïc sinh ra bôûi heä P höõu haïn, vaø luùc naøy ta noùi KGVT (V, R) laø höõu haïn chieàu vaø
- Coù chieàu laø p, kí hieäu dim(V)=p.
- KGVT sinh bôûi heä P laø KGVT bao goàm caùc phaàn töû laø toå hôïp tuyeán tính caùc phaàn töû cuûa P.
- Trong R^n, $E=\{(1,0,..,0),(0,1,..,0),..,(0,0,..,1)\}$ laø <u>cô sôû chính taéc</u>.

<u>*Ví du 0*</u>: Trong R^2 cho x=(1,2) thì $[x]_E = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

<u>*Ví du 1*</u>: E la khong gian nghiem cua:

$$\begin{cases} 3x_1 + 4x_2 + x_3 + 2x_4 + 3x_5 = 0 \\ 5x_1 + 7x_2 + x_3 + 3x_4 + 4x_5 = 0 \\ 4x_1 + 5x_2 + 2x_3 + x_4 + 5x_5 = 0 \\ 7x_1 + 10x_2 + x_3 + 6x_4 + 5x_5 = 0 \end{cases}$$

trong ví du tröôc la mot KGVT con cua R^5 co dim(E) = 2. va cô sô cua E la P=$\{(-3,2,1,0,0),(-5,3,0,0,1)\}$.

<u>*Ñònh ly:*</u> F, G la hai KGVT con höu han chieu cua V ta co:

$$\boxed{\dim(F+G) = \dim(F) + \dim(G) - \dim(F \cap G)}$$

<u>*Ví du 7:*</u> Trong R^4 cho u=(1,0,1,0), v=(0,1,-1,0), w=(1,1,1,1), x=(0,0,1,0), y=(1,1,0,-1), F= <u,v,w>, G= <x,y>.

a/ Tìm dimF, dimG.

b/ Tìm kgvt con sinh ra F∩G. Tìm dim(F∩G).

c/ Tìm kgvt con sinh ra F+G. Tö ño suy ra dim(F+G).

Giai:

Sinh vien Co the chöng minh F+G, F∩G la 2 khong gian con R^4.

a/ Vì u,v,w ÑLTT, nen dimF=3, va x, y ÑLTT, nen dimG=2

Ta se nghiem lai cong thöc: dim(F+G) = dim(F) + dim(G) - dim(F∩G).

b/ That ra tìm F∩G la giao 2 sieu phang au+bv+cw va dx+ey trong ño tìm a,b,c,d,e ∈R. Vay tö phöông trình: au+bv+cw = dx+ey.

$\Leftrightarrow au + bv + cw + d(-x) + e(-y) = 0 \Leftrightarrow$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \ldots \to \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Giai he nay ta co nghiem: a=2, b=2, c=-1, d=-1, e=1,

vay mot N^0 la: 2u+2v-w+x-y=0 hay 2u+2v-w= -x+y =(1,1,-1,-1)

==> <(1,1,-1,-1)> = F∩G, vay dim(F∩G)=1.

c/ Do menh tren ta co F+G= $\langle\langle u,v,w\rangle \cup \langle x,y\rangle\rangle = \langle u,v,w,x,y\rangle$ ta se chöng minh F+G= <u,v,w,x,y> = <u,v,w,x> = < u,v,w,y >. That vay

Vì x,y cung ÑLTT nen dim(G)=2. Nhöng cung co u,v,w,x ÑLTT nen dim(F+G)≥ 4, ma dim R^4=4, nen dim(F+G)= 4, va

F+G= <u,v,w,x,y> = <u,v,w,x> = < u,v,w,y > =R^4. ¢

<u>*Ví du 8:*</u> Trong R^4 cho hai he phöông trình:(I): $x_1 - x_2 - x_3 + x_4 = 0$, ñat khong gian N^0 (I) la F, va he (II) $\begin{cases} x_1 + x_4 = 0 \\ x_1 - x_2 = 0 \end{cases}$, ñat khong gian N^0 (II) la G, hay tìm mot cô sô cua $F \cap G$.

<u>Cach 1:</u> Ta de dang tìm ñöôc (tìm x_1 qua cac bien con lai)

F = $\langle f_1=(1,1,0,0), f_2=(1,0,1,0), f_3=(-1,0,0,1)\rangle$ hay bieán ñoåi ñeå coù:

F = $\langle u=(1,0,1,0), v=(0,1,-1,0), w=(1,1,1,1)\rangle$, trong ñoù:

$u = f_2 = (1,0,1,0), v = f_1 - f_2 = (0,1,-1,0), w = f_1 + f_2 + f_3 = (1,1,1,1)$, dimF=3,

G = $\langle x=(1,1,0,-1), y=(0,0,1,0)\rangle$, do N^0 laø: $(x_1,x_1,x_3,-x_1)$

dimG=2, Döõ lieäu naøy töông töï ví duï treân. Neân theo keát quaû treân ta coù: $F \cap G = \langle(1,1,-1,-1)\rangle$

<u>Caùch 2:</u> Ta gheùp hai phöông trình : $\begin{cases} x_1 - x_2 - x_3 + x_4 = 0 \\ x_1 + x_4 = 0 \\ x_1 - x_2 = 0 \end{cases}$

Ta deã daøng tìm ñöôïc ker ma traän: $F \cap G = \langle(-1,-1,1,1)\rangle$. ¢

<u>Ví duï15:</u> Tìm heä phöông trình tuyeán tính maø heä nghieäm laø truøng vôùi KGVT sinh bôûi 3 vectô α_1, α_2,α_3: α_1=(-1,0,1,2), α_2=(3,4,-2,5), α_3=(1,4,0,9),

Giaûi:

Goïi x=(x_1, x_2, x_3, x_4) $\in \langle a_1,a_2,a_3\rangle$,

coù $\exists y_i$, $x = y_1a_1 + y_2a_2 + y_3a_3 \Leftrightarrow \begin{cases} -y_1+3y_2+y_3 = x_1 \\ \quad +4y_2+4y_3 = x_2 \\ y_1-2y_2 \quad = x_3 \\ 2y_1+5y_2+9y_3 = x_4 \end{cases}$

$$[A|b] = \left[\begin{array}{ccc|c} -1 & 3 & 1 & x_1 \\ 0 & 4 & 4 & x_2 \\ 1 & -2 & 0 & x_3 \\ 2 & 5 & 9 & x_4 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 3 & 1 & x_1 \\ 0 & 4 & 4 & x_2 \\ 0 & 1 & 1 & x_1+x_3 \\ 0 & 11 & 11 & 2x_1+x_4 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|c} -1 & 3 & 1 & x_1 \\ 0 & 1 & 1 & x_1+x_3 \\ 0 & 0 & 0 & -4x_1+x_2-4x_3 \\ 0 & 0 & 0 & -9x_1-11x_3+x_4 \end{array}\right]$$ ==> r(A)=r(A,b)=2<3 ,

ñaùp soá : $\begin{cases} -4x_1+x_2-4x_3 \quad = 0 \\ -9x_1 \quad -11x_3+x_4 = 0 \end{cases}$, nghieäm laø $a_3 = 2a_1 + a_2$

<u>Ví duï16:</u> Cho hai heä phöông trình:(I) $2x_1+2x_2+x_3+4x_4=0$, ñaët khoâng gian N^0 (I) laø F, vaø heä (II) $\begin{cases} 0x_1+9x_2+1x_3+8x_4=0 \\ 1x_1-2x_2+0x_3-1x_4=0 \end{cases}$, ñaët khoâng gian N^0 (II) laø G, haõy tìm moät cô sôû cuûa $F \cap G$.

<u>Caùch 1:</u> Ta deã daøng tìm ñöôïc, tìm x_3 qua caùc bieán coøn laïi:

F = $\langle f_1=(1,0,-2,0), f_2=(0,1,-2,0), f_3=(0,0,-4,1)\rangle$, dimF=3

Tìm x_1, x_2, qua caùc bieán coøn laïi:

G = $\langle g_1=(0,1,7,-2), g_2=(1,0,-8,1)\rangle$ dimG=2, dimF+G=4

==> dim $F \cap G$ = dim(F)+ dim(G)- dim(F+G)=3+2-4=1.

Ngoaøi ra ta coù: $-3f_1+2f_2+1.f_3 = 2g_1+1.g_2 =$ (1, 2, 6, -3) vaäy <(1, 2, 6, -3)>= $F \cap G$

Caùch 2: Ta deã daøng tìm ñöôïc, tìm x_3 qua caùc bieán coøn laïi:

F = $\left\langle t_1 = (1,0,-2,0), t_2 = (0,1,-2,0), t_3 = (0,0,-4,1) \right\rangle$, hay

F= $\left\langle f_1 = t_1,\ f_2 = -t_2,\ f_3 = t_1 + t_2 - t_3 = (1,1,0,-1) \right\rangle$, dimF=3

Tìm x_3 qua caùc bieán coøn laïi:

G = $\left\langle g_1 = (2,1,-9,0), v = (1,0,-8,1) \right\rangle = \left\langle g_1,\ g_2 = (-1,-1,1,1) = v - g_1 \right\rangle$,

Vaø ta coù: $-2f_1 + 1f_2 + 3f_3 = -g_1 - 3g_2$ vaäy <(1, 2, 6, -3)>= $F \cap G$

Caùch 3: Ta goäp hai phöông trình laïi: $\begin{cases} 2x_1 + 2x_2 + x_3 + 4x_4 = 0 \\ 0x_1 + 9x_2 + 1x_3 + 8x_4 = 0 \\ 1x_1 - 2x_2 + 0x_3 - 1x_4 = 0 \end{cases}$,

Ta deã daøng tìm ñöôïc ker cuûa ma traän naøy laø: (1, 2, 6, -3)

==> $F \cap G = \left\langle fg = (1,2,6,-3) \right\rangle$. ¢

2.2 Chuyeån ñoåi cô sôû:

Baây giôø coù coù hai cô sôû ñöôïc saép B vaø $B^/$ cuûa KGVT V coù chieàu laø n, caùc toaï ñoä trong cô sôû B vaø toaï ñoä trong cô sôû $B^/$ lieân heä vôùi nhau ra sao ?

Cuï theå $B = \{u_1, u_2, \ldots, u_n\}$, $B^/ = \{u_1^/, u_2^/, \ldots, u_n^/\}$. Ta goïi P∈ M_n(R) laø ma traän chuyeån cô sôû töø B sang $B^/$, kí hieäu P=(B→$B^/$) neáu P ñöôïc thaønh laäp:

$P = \left[[u_1^/]_B, [u_2^/]_B, \ldots, [u_n^/]_B \right] = P_{B \to B^/}$, noùi caùch khaùc P laø ma traän coät toïa ñoä cuûa caùc vectô cô sôû $B^/$ ñöôïc bieåu dieãn qua cô sôû B.

Meänh ñeà: Vôùi caùc giaû thieát ôû treân ta coù caùc khaúng ñònh sau:

i/ (B→B) =I_n.

ii/ (B→$B^{//}$) = (B→$B^/$) × ($B^/$→$B^{//}$).

iii/ (B→$B^/$) = ($B^/$→B)$^{-1}$.

Ví duï1:

Trong R^2, vôùi cô sôû $B^/ = \{\vec{e}_1^{\,/}, \vec{e}_2^{\,/}\} = \{(2,1),(-1,1)\}$, x coù toaï ñoä $[x]_{B^/}$ =$(1,1)^T$. Xeùt cô sôû khaùc: $B = \{\vec{e}_1, \vec{e}_2\} = \{(0,1),(1,0)\}$. Tìm toaï ñoä môùi $[x]_B$ cuûa x trong cô sôû B.

sinh vieân haõy chöùng minh $B^/$, B ôû ñaây laø 2 cô sôû cuûa R^2.

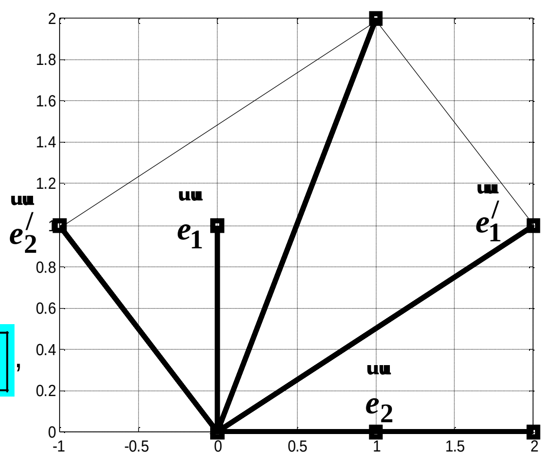
Do x=1$\vec{e}_1^{\,/}$ +1. $\vec{e}_2^{\,/}$ = $\vec{e}_1^{\,/}$ + $\vec{e}_2^{\,/}$ =

=(1$\vec{e}_1$ +2. $\vec{e}_2$) + (1$\vec{e}_1$ -1. $\vec{e}_2$)=

=(2$\vec{e}_1$ +1. $\vec{e}_2$), vaäy toaï ñoä $[x]_B$ cuûa x trong cô sôû B laø $(2,1)^T$.

Caùch khaùc: $\boxed{[x]_B = P_{B \to B^/} [x]_{B^/}}$,

$P_{B \to B^/} = P_{B \to E}.P_{E \to B^/}$



$$P_{B\to E} = P_{E\to B}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

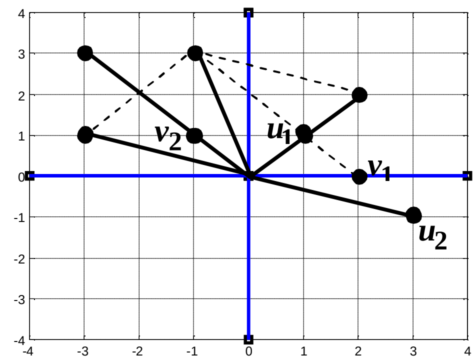
$$[x]_B = P_{B\to B'}\,[x]_{B'} = P_{E\to B}^{-1} P_{E\to B'}\,[x]_{B'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Ví duï4: Trong R^2, vôùi cô sôû $U = \{\vec{u_1}, \vec{u_2}\} = \{(1,1),(3,-1)\}$,

x coù toaï ñoä $[x]_U = (2, -1)^T$. Xeùt cô sôû khaùc:

$V = \{\vec{v_1}, \vec{v_2}\} = \{(2,0),(-1,1)\}$.

Tìm toaï ñoä môùi cuûa $[x]_V$ trong cô sôû V.



Caùch 3: Ta laáy cô sôû chính taéc E laøm cô sôû trung gian.

$P_{E\to U} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$, $P_{E\to V} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$,

$P_{V\to U} = P_{V\to E} \cdot P_{E\to U} = (P_{E\to V})^{-1} \cdot P_{E\to U}$ ==>

$$\boxed{[x]_V = P_{V\to U}\,[x]_U} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

¢

Ví duï5:

Trong R^3, vôùi cô sôû $U = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \{(1,0,1),(1,3,-1),(0,1,2)\}$, x coù toaï ñoä $[x]_U = (2, -1, 5)^T$. Xeùt cô sôû khaùc: $V = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \{(1,2,0),(0,1,-1),(-1,0,1)\}$.

Tìm toaï ñoä môùi cuûa x trong cô sôû V.

Caùch 2: Ta laáy cô sôû chính taéc E laøm cô sôû trung gian.

Ta laáy cô sôû trung gian laø cô sôû chính taéc:

$E = \{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \{(1,0,0),(0,1,0),(0,0,1)\}$. Baèng ngoân ngöõ ma traän ta coù:

Ma traän chuyeån töø cô sôû E $\to$ V laø ma traän: $P_{E\to V} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

Ma traän chuyeån töø cô sôû E $\to$ U laø ma traän: $P_{E\to U} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

Vaäy ma traän chuyeån cô sôû V $\to$ U laø tích hai ma traän:

$$P_{V\to U} = P_{V\to E}\, P_{E\to U} = P_{E\to V}^{-1}\, P_{E\to U} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix} =$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 1 & 1 \\ \frac{-4}{3} & 1 & -1 \\ \frac{-1}{3} & 0 & 1 \end{bmatrix}.$$

Vaäy toaï ñoä $[x]_V$ trong cô sôû môùi V laø:

$$[x]_V = P_{V \to U}[x]_U = \begin{bmatrix} \frac{2}{3} & 1 & 1 \\ \frac{-4}{3} & 1 & -1 \\ \frac{-1}{3} & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{16}{3} \\ \frac{-26}{3} \\ \frac{13}{3} \end{bmatrix} \quad ¢$$

Phan 3:

CHÖÔNG 1: BIEN NGAU NHIEN (BNN)

I. Bien ngau nhien

Ñònh nghóa: Bien ngau nhien (bnn) (con goi la ñai löông ngau nhien).
Anh xa X tö khong gian mau Ω vao tap so thöc R:

$$X : \Omega \to R$$
$$\omega \in X, \omega \mapsto X(\omega) \in R$$

ñöôc goi la bnn. Thöông ñöôc ky hieu la: X,Y,Z....

- Bien ngau nhien rôi rac (BNNRR): Neu $X(\Omega)$ la höu han hay vo han ñem ñöôc.
- Bien ngau nhien lien tuc (BNNLT): Neu $X(\Omega)$ la mot khoang hay mot so khoang hay toan bo R.

Ví du1: Ban lien tiep n vien ñan ñoc lap vao bia. Goi X la so vien ñan trung ñích $\Rightarrow X = \{0,1,2,...,n\}$, la bnn rôi rac.

Ví du2: Tuoi tho cua mot thiet bò ta bat ñau khao sat la bnn:

$$\Omega = \{t : 0 \le t < \infty\} = [0, \infty) \Rightarrow X(t) = t,$$

la bnn lien tuc.

II. Luat phan phoi xac suat cua bien ngau nhien

2.1 Bang phan phoi xac suat cua bnn rôi rac

Ta co bang phan phoi sau, vôi ñieu kien $\sum P_i = 1$

X	x_1	x_2	...	x_j	...	x_n
P(X)	p_1	p_2	...	p_j	...	p_n

Ví du1: Tung 2 con suc xac, X bnn la tong 2 mat, lap bang phan phoi xac suat

Giai

De dang lap ñöôc bang phan phoi xac suat nhö sau: $X = \overline{2,12}$

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{6^2}$	$\frac{2}{6^2}$	$\frac{3}{6^2}$	$\frac{4}{6^2}$	$\frac{5}{6^2}$	$\frac{6}{6^2}$	$\frac{5}{6^2}$	$\frac{4}{6^2}$	$\frac{3}{6^2}$	$\frac{2}{6^2}$	$\frac{1}{6^2}$

Ví du2: Trong 10 san pham co 6 chính pham, lay ngau nhien khong hoan lai ra 2 san pham, lap bang phan phoi xac suat cua so chính pham ñöôc lay ra.

Giai

Goi X la so chính pham ñöôc lay ra, ta co: X = 0,1,2

$$P(X = i) = \frac{C_6^i C_4^{2-i}}{C_{10}^2} \Rightarrow P(0) = \frac{2}{15},\ P(1) = \frac{8}{15},\ P(2) = \frac{5}{15}$$

X	0	1	2
P(X)	$\frac{2}{15}$	$\frac{8}{15}$	$\frac{5}{15}$

Chu y: Trong hai ví du tren ta luon co: $\sum P_i = 1$

III. Ham phan phoi (hpp)

Ñònh nghóa: Ham phan phoi (hpp).

$$F_X(x) = F(x) = P(X < x) = P(w : X(w) < x), x \in R$$

ñöôc goi la ham phan phoi cua X.

Nhan xet: Gia trò cua ham F tai x, thì bang tong cac xac suat cua cac X ô tröôc x, nen $F(x)$ con ñöôc ñònh nghóa ro rang hôn nhö sau:

$$F(x) = \sum_{x_j < x} P(x_j < x)$$

<u>Tính chat:</u> Ta co the chöng minh cac tính chat sau:

Tính chat 1: $0 \le F(x) \le 1$

Tính chat 2: Ham phan phoi luon la ham khong giam, töc

$x_1 < x_2 \Rightarrow F(x_1) \le F(x_2)$

Tính chat 3: $\lim\limits_{x \to +\infty} F(x) = 1$, $\lim\limits_{x \to -\infty} F(x) = 0$

Tính chat 4: $P(a \le X \le b) = F(b) - F(a)$

Tính chat 5: Xac suat ñe <u>bnnlt</u> X nhan gia trò tai mot ñiem luon bang 0,

do: $P(X = a) = P(a \le X \le a) = F(a) - F(a) = 0$

Tính chat 6: Neu X la bnn lien tuc thì:

$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$

Nhan xet: Xac suat tai mot ñiem chæ co nghóa vôi bnn rôi rac, con xac suat tai mot ñiem vôi bnn lien tuc luon bang 0.

3.1 Bien ngau nhien rôi rac (BNNRR)

Neu ta co bang phan phoi sau:

X	x_1	x_2	x_j	...	x_n
P(X)	p_1	p_2	p_j	...	p_n

Gia sö $x_1 < x_2 < ... < x_n$, theo ñònh nghóa ô tren, ta co ham phan phoi cho BNN rôi rac trong tröông hôp nay la:

$$F(x) = \begin{cases} 0, & x < x_1 \\ p_1, & x_1 \le x < x_2 \\ p_1 + p_2, & x_2 \le x < x_3 \\ \dots\dots\dots\dots\dots\dots & \\ p_1 + p_2 + ... + p_{n-1}, & x_{n-1} \le x < x_n \\ p_1 + p_2 + ... + p_{n-1} + p_n = 1, & x_n \le x < \infty \end{cases}$$

Tai x bat ky ta co: $F(x) = \sum_{x_j < x} p_j = \sum_{x_j < x} P(x_j < x)$

ta co the kiem chöng cac bieu thöc sau cho BNNRR:

$P(a < X < b) = F(b - 0) - F(a)$

$P(a \le X \le b) = F(b) - F(a - 0)$

$P(a \le X < b) = F(b - 0) - F(a - 0)$

<u>Ví du1:</u> Mot lo sp co 5 sp tot va 4 sp xau. Lay ngau nhien tö lo ra 3 sp. Goi X la so sp tot trong 3 sp lay ra. Tìm phan phoi xac suat cua X, viet ham phan phoi va tính $P(1 \le X < 3)$

<u>Giai</u>

Ta coù $X = \{0;1;2;3\}$. $P[X = k] = \dfrac{C_5^k C_4^{3-k}}{C_9^3}$, $k = 0;1;2;3$

X	0	1	2	3
P(X)	$\frac{2}{42}$	$\frac{15}{42}$	$\frac{20}{42}$	$\frac{5}{42}$

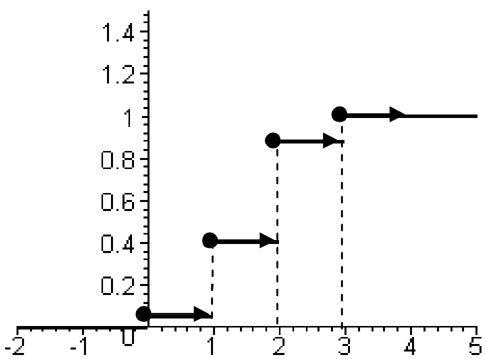
ta coäng doàn xaùc suaát theo töøng khoaûng ñeå coù haøm phaân phoái xaùc suaát nhö sau:

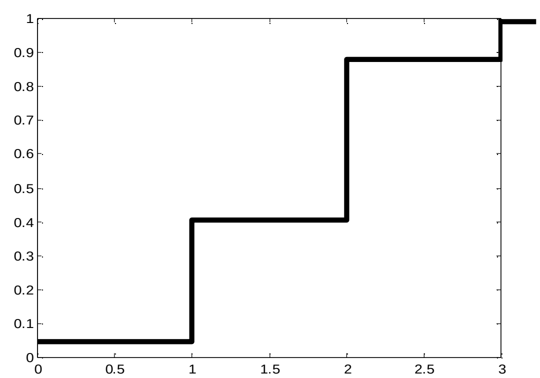
$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{2}{42}, & 0 \le x < 1 \\ \dfrac{17}{42}, & 1 \le x < 2 \\ \dfrac{37}{42}, & 2 \le x < 3 \\ 1, & 3 \le x < \infty \end{cases}$$

, Vôùi coâng thöùc ôû treân cho BNNRR ta coù:

$$P(1 \le X < 3) = F(3-0) - F(1-0) = \frac{37}{42} - \frac{2}{42} = \frac{35}{42} = P(1) + P(2) = \frac{15}{42} + \frac{20}{42}$$

Haøm phaân phoái coù ñoà thò laø haøm hình thang taêng.



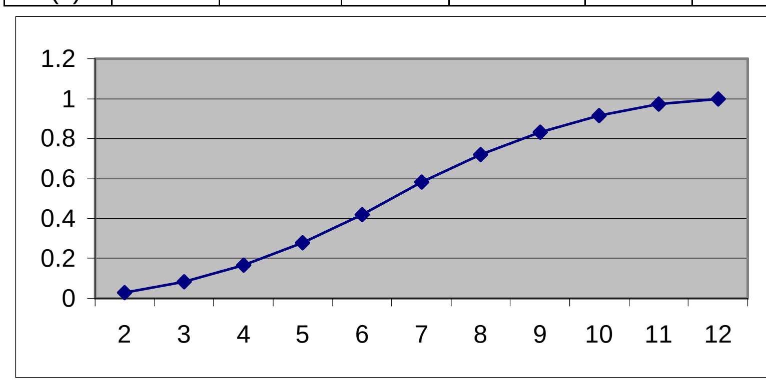


<u>Ví duï2:</u> Tung 2 con suùc xaéc, X bnn laø toång 2 maët, laäp baûng phaân phoái xaùc suaát, tìm haøm phaân phoái.

<u>Giaûi</u>

Ta ñaõ xeùt ví duï naøy, ôû ñaây ta tìm haøm F(x) theo caùch coäng doàn.

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36
F(x)	1/36	1/12	1/6	5/18	5/12	7/12	13/18	5/6	11/12	35/36	1



Ñaây laø ñoà thò minh hoïa haøm F(x)

3.2 Bieán ngaãu nhieân lieân tuïc (BNNLT)

Töông öng vôi bang phan phoi xac suat cua bnn rôi rac thì

vôi bnn lien tuc ta co khai niem <u>ham mat ño xac suat</u>

Ñònh nghóa: X nhan cac gia trò lap ñay (a; b) (a, b co the vo han). Öng vôi moi $x \in (a;b)$, mat ño xac suat tai x ky hieu la $f(x) \ge 0$ va $\int_{-\infty}^{+\infty} f(x)dx = \int_a^b f(x)dx = 1$.

Giong nhö rôi rac, bien nn lien tuc cung ñöôc ñònh nghóa:

$\boxed{F(x) = \int_{-\infty}^{x} f(t)dt = P(X < x)}$ <---> $F(x) = \sum_{x_j < x} P(x_j < x)$

Do ñònh nghóa nhö the nen ta co cac ket qua sau:

$$\boxed{P[a \le X \le b] = \int_a^b f(x)dx = F(b) - F(a)}, \ (a \le a < b \le b)$$

$$\boxed{\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\int_{-\infty}^{x} f(t)dt\right) = (F(x))'_x = f(x)}$$

· Y nghóa ham mat ño: $P(x \le X \le x + \Delta x) \approx f(x)\Delta x$

Töc xac suat ñe X rôi trong khoang $[x, x + \Delta x]$ gan nhö tæ le vôi f(x), nen no ñöôc goi la ham mat ño.

<u>Ví du1:</u> Ham mat ño xac suat: $f(x) = \begin{cases} 0, & x < 0 \\ 4x, & 0 \le x < \frac{1}{2} \\ 4 - 4x, & \frac{1}{2} \le x < 1 \\ 0, & 1 \le x < \infty \end{cases}$,

a/ Chöng minh f(x) la ham mat ño.

b/ Tìm ham phan phoi xac suat. Tính $P\left(\frac{1}{4} < X < \frac{3}{5}\right)$

<u>Giai</u>

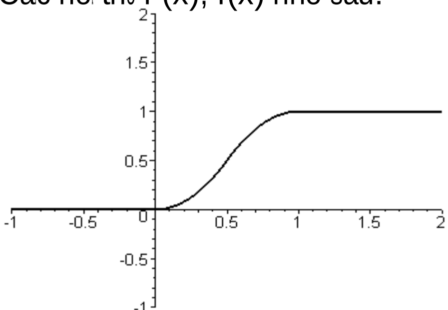
a/ Sv tö kiem chöng $\int_{-\infty}^{+\infty} f(x)dx = 1$.

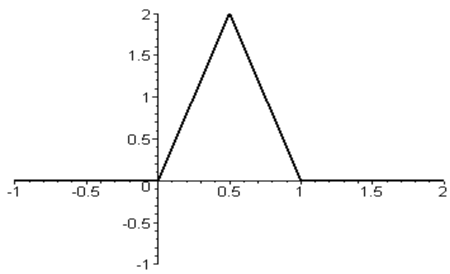
b/ Ta cong don tích phan ñe co:

$$F(x)=\int_{-\infty}^{x} f(t)\,dt=\begin{cases} 0=\int_{-\infty}^{0} 0\,dt, & x<0 \\ 2x^2=\int_{0}^{x} 4t\,dt, & 0\le x<\frac{1}{2} \\ 4x-2x^2-1=\int_{0}^{\frac{1}{2}} 4t\,dt+\int_{\frac{1}{2}}^{x}(4-4t)\,dt, & \frac{1}{2}\le x<1 \\ 1=\int_{-\infty}^{\infty} f(t)\,dt=\int_{0}^{\frac{1}{2}} 4t\,dt+\int_{\frac{1}{2}}^{1}(4-4t)\,dt, & 1\le x<\infty \end{cases}$$

Ta co: $P\left(\frac{1}{4}<X<\frac{3}{5}\right)=F\left(\frac{3}{5}\right)-F\left(\frac{1}{4}\right)=\frac{17}{25}-\frac{1}{8}=\frac{111}{200}$.

Cac ño thò F(x), f(x) nhö sau:





Ham mat ño con co the mô rong la: khong can lien tuc.

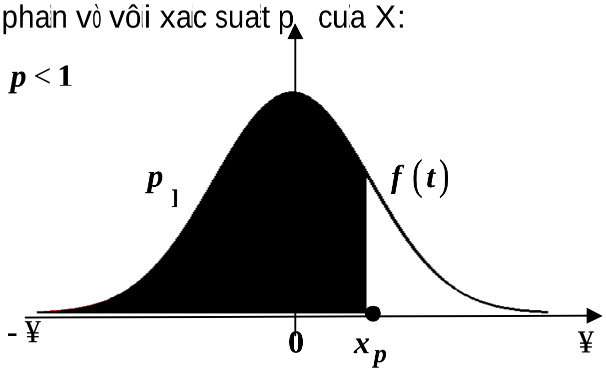
IV. Cac ñac tröng so

4.1 Ñònh nghóa: Ñiem phan vò, Median (med(X)).

Hoanh ño x_p ñöôc goi la ñiem phan vò vôi xac suat p, cua X:

$$x_p=\min\{x : F_X(x)\ge p\},\quad 0<p<1$$

$$\Leftrightarrow \int_{-\infty}^{x_p} f(t)\,dt=p$$



Ñoi vôi bnn lien tuc:
trong hình ve la ño thò
cua ham mat ño, dien tích
boi ñen bang p, tai hoanh ño x_p.

Ñac biet $p=\frac{1}{2}$ ñöôc goi la median ([Med], $x_{1/2}$ la ñiem trung vò cua X:hay noi cach khac:

$$x_{1/2}=med(X)\Leftrightarrow \int_{-\infty}^{x_{1/2}} f(t)\,dt=\frac{1}{2}$$

Ñoi vôi bnn rôi rac thì trung vò la 1 gia trò cua X ma tai ño <u>xac suat ñööc chia ñeu töông ñoi hai ben</u>, nghóa la $med(X) = x_i$

$F\left[x_i\right] = P\left[X < x_i\right] \le \frac{1}{2} \le F\left[x_{i+1}\right] = P\left[X < x_{i+1}\right]$. Thuat toan tìm

$med(X) = x_i$ nhö sau: cong don xac suat, bat ñau tö p_1, p_2 ... ñen chæ so i cua x_i sao cho: $\sum_{j=1}^{i} p_j \ge \frac{1}{2}$

<u>Ví du1:</u> Cho X rôi rac co luat phan phoi

X	1	2	3	4	5
P(X)	0,1	0,2	0,15	0,1	0,45

$Med[X] = 4$.

Do tong xac suat ô hai ben cua X=4 ñeu bang 0,45.

Trong excel: Median(range). Range co the gom cac cell trong

<u>Ví du2:</u> Cho X BNN rôi rac co luat phan phoi

X	0	1	$\boxed{2}$	3
P(X)	$\frac{2}{42}$	$\frac{15}{42}$	$\frac{20}{42}$	$\frac{5}{42}$

$Med[X] = 2$, vì F(x=2) < $\frac{2}{42} + \frac{15}{42} < \frac{1}{2}$, F(x=3)= $\frac{2}{42} + \frac{15}{42} + \frac{20}{42} > \frac{1}{2}$.

4.2 Mode(X) (mot X).

Mode(X) cua BNN X la gia trò cua X=x sao cho xac suat tai X=x la lôn nhat <u>(xac xuat tin chac nhat)</u>

Trong excel: Mode (range). Range co the ca cell trong

<u>Ví du1:</u> Cho X BNN rôi rac co luat phan phoi ô ví du tren:

X	0	1	2	3
P(X)	$\frac{2}{42}$	$\frac{15}{42}$	$\frac{20}{42}$	$\frac{5}{42}$

thì ta co: mode(X) =2.

<u>Ví du2:</u> Cho X BNN lien tuc co ham mat ño nhö ño thò cua phan ñònh nghia med(X) ô tren, ta co mode(X) =10.

4.3 Ky vong toan, moment cap k, phöông sai.

· Ñònh nghóa: Ky vong thöông kí hieu: $E(X) = \overline{X} = M(X) = \mu(X) = m_X$.

E: Expectation, M: Mean, nen ky vong con ñööc goi la gia trò trung bình.

+Vôi X rôi rac ta ñònh nghóa:

X	x_1	x_2	x_j	...	x_n
P(X)	p_1	p_2	p_j	...	p_n

$$E(X) = M(X) = \mu(X) = x_1 p_1 + x_2 p_2 + ... + x_n p_n = \sum_{i=1}^{n} x_i p\left(x_i\right).$$

+ Neu X lien tuc thì $E(X) = M(X) = \mu(X) = \mu_X = \int_{-\infty}^{+\infty} x f(x) dx$.

Trong excel: Average (range). Range co the gom cac cell trong

Ví du: Trong bình ñöng 10 qua cau gioing nhau nhöng khac trong löông gom 5 qua co trong löông nang 1kg, 2 qua co trong löông 2kg va 3 qua co trong löông 3 kg. Lay ngau nhien 1 qua, goi X la trong löông qua cau ño. X co luat phan phoi:

X	1kg	2kg	3kg
P(X)	0,5	0,2	0,3

Suy ra E(X) = $1 \times \frac{5}{10} + 2 \times \frac{2}{10} + 3 \times \frac{3}{10} = \frac{1 \times 5 + 2 \times 2 + 3 \times 3}{10}$ = 1,8kg.

(trong löông trung bình cua 1 qua cau theo y nghóa thong thöông cua bình quan gia quyen trong ñôi song thöc te).

· Y nghóa ky vong:

Ky vong la gia trò trung bình co trong so theo xac suat cua ñai löông ngau nhien X, la trung tam ñiem cua phan phoi ma cac gia trò cu the cua X se tap trung quanh ño.

· Moment cap k:

Rôi rac: $E(X^k) = \sum_{i=1}^{n} x_i^k p\left(x_i\right)$

Lien tuc: $E\left(X^k\right) = \int_{-\infty}^{\infty} x^k f(x) dx$, neu tích phan nay hoi tu.

¨ Moment tam cap k:

Rôi rac: $E\left(\left[X - EX\right]^k\right) = E\left(\left[X - \mu\right]^k\right) = \sum_{i=1}^{n} \left[x_i - EX\right]^k p\left(x_i\right)$

Lien tuc: $E(X - EX)^k = E(X - \mu)^k = \int_{-\infty}^{\infty} (x - \mu)^k f(x) dx$,

neu tích phan nay hoi tu.

Tröông hôp rieng: phöông sai la moment tam cap k= 2:

¨ Phöông sai: (kí hieu: var(X) = D(X) = $\sigma^2(X) = \sigma_X^2$)

Rôi rac: $varX = E\left(\left[X - EX\right]^2\right) = E\left(\left[X - \mu\right]^2\right) = \sum_{i=1}^{n} \left[x_i - \mu\right]^2 p\left(x_i\right)$

Lien tuc:

$$\mathrm{var}(X) = D(X) = \sigma^2(X) = E(X - EX)^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Y nghóa phöông sai

Phöông sai la sai so bình phöông trung bình cua ñai löông ngau nhien X so vôi trung tam ñiem ky vong. Phöông sai dung ñe ño möc ño phan tan cua X quanh ky vong.

¨ Ño lech tieu chuan: $\boxed{\sqrt{\sigma^2(X)} = \sqrt{\mathrm{var}(X)} = \sigma_X = se(X) =_x \sigma_n}$

Trong excel: StDevP (range). Range co the gom cac cell trong

Menh ñe:

1/ $E(bX + cY + d) = bE(X) + cE(Y) + d, \quad b,c,d \in R$.

2/ X, Y la BNN ñoc lap thì $\longrightarrow \boxed{E(XY) = E(X)E(Y)}$

3/ $\mathrm{var}(X) = E(X^2) - [E(X)]^2$, Khi ño

$$\mathrm{var}(X) = \sum_{j=1}^{n} x_j^2 p_j - \left(\sum_{j=1}^{n} x_j p_j\right)^2$$, X rôi rac.

$$\mathrm{var}(X) = \int_{-\infty}^{+\infty} x^2.f(x)dx - \left(\int_{-\infty}^{+\infty} x.f(x)dx\right)^2$$, X lien tuc.

4/ var(C) = 0

5/ var(X) = 0 suy ra X={c} töc P[X = c] = 1.

6/ X, Y la BNN ñoc lap thì $\mathrm{var}(X \pm Y) = \mathrm{var}(X) + \mathrm{var}(Y)$

7/ Tong quat hôn 6/ ta co: $\mathrm{var}(bX \pm cY) = b^2\,\mathrm{var}(X) + c^2\,\mathrm{var}(Y)$

8/ Neu $Y = \varphi[x] \Rightarrow E(Y) = E(\varphi[x]) = \begin{cases} \sum_i \varphi[x_i]\, p_i \\ \int_{x=-\infty}^{x=\infty} \varphi[x]\, f(x)dx \end{cases}$

$$\mathrm{var}(\varphi[x]) = \sum_{j=1}^{n} \varphi^2[x_j]\, p_j - \left(\sum_{j=1}^{n} \varphi[x_j]\, p_j\right)^2$$, X rôi rac.

$$\mathrm{var}(\varphi[x]) = \int_{-\infty}^{+\infty} \varphi^2[x].f(x)dx - \left(\int_{-\infty}^{+\infty} \varphi[x].f(x)dx\right)^2$$, X lien tuc.

Ví du3: BNN X co ham mat ño xac suat: $f(x) = \begin{cases} \dfrac{k}{x^2}, & 1 \le x \le 3 \\ 0, & con\ lai \end{cases}$

a/ Tìm k. b/ Tìm E(X) c/ Tìm Med(X)

d/ Tìm ky vong cua $Y = X^3 + \dfrac{1}{X}$

Giai:

a/ Döa vao $\int_{x=-\infty}^{x=\infty} f(x)dx = \int_{x=1}^{x=3} f(x)dx = 1 \Rightarrow$ k=$\dfrac{3}{2}$.

b/ $E(X)=\int_{x=1}^{x=3} xf(x)dx=\int_{x=1}^{x=3} x\frac{3}{2x^2}dx=\int_{x=1}^{x=3}\frac{3}{2x}dx=\frac{3}{2}\ln(3)$

c/ Med(X)=x, theo ñònh nghóa ta coù:

$$F(x)=\int_{t=1}^{t=x} f(t)dt=\frac{1}{2}\Leftrightarrow\int_{t=1}^{t=x}\frac{3}{2t^2}dt=\frac{1}{2}\Leftrightarrow\frac{3}{2}\frac{x-1}{x}=\frac{1}{2}\Leftrightarrow x=\frac{3}{2}$$

d/ $E(\varphi[x])=\int_{x=1}^{x=3}\varphi[x]f(x)dx=\int_{x=1}^{x=3}\left(x^3+\frac{1}{x}\right)\frac{3}{2x^2}dx=$

$$=\frac{3}{2}\int_{x=1}^{x=3}\left(x+\frac{1}{x^3}\right)dx=\frac{20}{3}$$ ¢

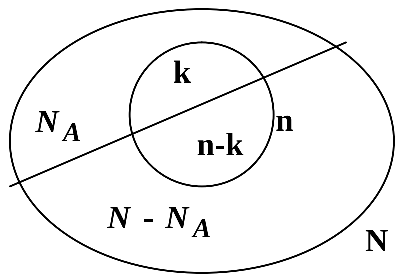
5.2 Phaân phoái sieâu boäi $X\in H(N,N_A,n)$

Xeùt taäp coù N phaàn töû, trong ñoù coù N_A phaàn töû coù tính chaát A. Töø taäp ñoù laáy ra n phaàn töû. Goïi X laø soá phaàn töû coù tính chaát A thì X coù phaân phoái sieâu boäi.

Ta kí hieäu $X\in H(N,N_A,n)$,

moät vaøi taøi lieäu coøn kí hieäu

$X\sim H(N,N_A,n)$.



Ñònh nghóa

Phaân phoái sieâu boäi laø phaân phoái cuûa BNN rôøi raïc treân taäp X = {0; 1; 2; ...; n} vôùi xaùc suaát töông öùng laø

$p_k=P[X=k]=\frac{C_{N_A}^k C_{N-N_A}^{n-k}}{C_N^n}$, (chuù yù: neáu $k>N_A$ thì $p_k=0$)

5.3 Phaân phoái nhò thöùc $X\in B(n,p)$ hay $X\sim B(n,p)$

Daõy pheùp thöû Bernoulli laø daõy n pheùp thöû thoûa 3 ñieàu kieän

i/ Caùc pheùp thöû ñoäc laäp vôùi nhau.

ii/ Trong moãi pheùp thöû ta chæ quan taâm ñeán moät bieán coá A.

iii/ Trong moãi pheùp thöû xaùc suaát thaønh coâng luoân laø haèng soá

$$P(A)=p\Rightarrow P(\overline{A})=1-p=q,\ (0<p<1)$$

Ñònh nghóa

Phaân phoái nhò thöùc laø phaân phoái cuûa ñaïi löôïng ngaãu nhieân rôøi raïc treân taäp X = {0; 1; 2; ...; n} vôùi xaùc suaát thaønh coâng moãi laàn thöû laø p, thì xaùc suaát coù k laàn thaønh coâng laø:

$$p_k=P[X=k]=B(k,n,p)=C_n^k p^k q^{n-k}$$

Ví duï 2: Sau khi saûn xuaát xong, saûn phaåm cuûa nhaø maùy Ñaïi Hung ñöôïc ñoùng thaønh hoäp, moãi hoäp coù 10 saûn phaåm. Cho bieát soá saûn phaåm loaïi 1 trong moãi hoäp coù phaân phoái nhö sau:

Soá saûn phaåm loaïi 1	7	8	9	10
Tyû leä hoäp töông öùng	0.1	0.3	0.4	0.2

Moät khaùch haøng muoán mua moät loâ haøng 500 hoäp cuûa nhaø maùy. Khaùch haøng naøy kieåm tra töøng hoäp, baèng caùch choïn ngaãu nhieân 3 saûn phaåm trong hoäp, neáu caû 3 saûn phaåm loaïi 1 thì nhaän hoäp ñoù.

a) Tính xaùc suaát ñeå moät hoäp ñöôïc chaáp nhaän.

b) Tìm soá hoäp tin chaéc nhaát maø khaùch haøng coù theå nhaän ñöôïc trong 500 hoäp caàn muoán mua ôû treân.

Giaûi:

Vôùi giaû thieát coù 4 loaïi hoäp:

Soá saûn phaåm loaïi 1	7sp	8 sp	9 sp	10 sp
Tyû leä hoäp töông öùng	0.1	0.3	0.4	0.2

ñaây laø XS ñaày ñuû, neân ta duøng coâng thöùc xaùc suaát ñaày ñuû. Goïi:

A_7: "BC hoäp coù 7 sp loaïi 1", $P(A_7)=0.1$

A_8: "BC hoäp coù 8 sp loaïi 1", $P(A_8)=0.3$

A_9: "BC hoäp coù 9 sp loaïi 1", $P(A_9)=0.4$

A_{10}: "BC hoäp coù 10 sp loaïi 1", $P(A_{10})=0.2$

Goïi A: "BC chaáp nhaän hoäp ôû moãi laàn kieåm tra".

$$p=P(A)=P(A_7)P\left(A/A_7\right)+P(A_8)P\left(A/A_8\right)+P(A_9)P\left(A/A_9\right)+$$

$$+P(A_{10})P\left(A/A_{10}\right)$$

$$P\left(A/A_7\right)=\frac{C_7^3C_3^0}{C_{10}^3}=\frac{7}{24},\ P\left(A/A_8\right)=\frac{C_8^3C_2^0}{C_{10}^3}=\frac{7}{15},\ P\left(A/A_9\right)=\frac{C_9^3C_1^0}{C_{10}^3}=\frac{7}{10},\ P\left(A/A_{10}\right)=\frac{C_{10}^3C_0^0}{C_{10}^3}=1$$

$p=P(A)=0.6492$ ==> q= 0.3508.

Goïi X: BNN soá hoäp ñöôïc chaáp nhaän sau khi kieåm tra 500 hoäp, $X\in B(500,p=0.6492)$

goïi k laø soá hoäp tin chaéc nhaát nhaän ñöôïc, phaûi thoûa:

$np-q\le k\le np-q+1$. ta coù:

np= $500\times 0.6492=324.583$

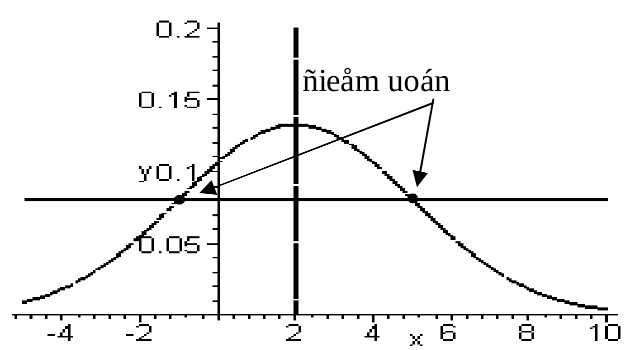
$np-q=500\times 0.6492-0.3508=324.232$

$np-q+1=324.232+1=325.232$, vaäy k=325 hoäp. ¢

6.6. Phaân phoái chuaån: $X\in N(\mu,\sigma^2),\sigma>0,\mu=const$.

Phaân phoái cuûa ñaïi löôïng ngaãu nhieân lieân tuïc X nhaän giaù trò treân R vôùi haøm maät ñoä phaân phoái: $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

Chuù yù: $\int_{-\infty}^{\infty}f(x)dx==\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=1$



döïa vaøo $\int_{-\infty}^{\infty}e^{-t^2}dx=\sqrt{\pi}$

Vôùi $\mu=2,\sigma=3$

ta coù ñoà thò nhö treân,

Tong quat ta co:

Ñiem cöc ñai $M = \left(\mu, \dfrac{1}{\sigma\sqrt{2\pi}}\right)$

va 2 ñiem uon: $U_1 = \left(\mu - \sigma, \dfrac{1}{\sigma\sqrt{2\pi e}}\right), U_2 = \left(\mu + \sigma, \dfrac{1}{\sigma\sqrt{2\pi e}}\right)$

6.6.1. Phan phoi chuan ñôn gian:

Ñau tien ta co: $X \in N(\mu, \sigma^2) \Rightarrow T = \dfrac{X - \mu}{\sigma} \in N(0,1)$, ta se chöng minh ô phan VIII (ham cua BNN).

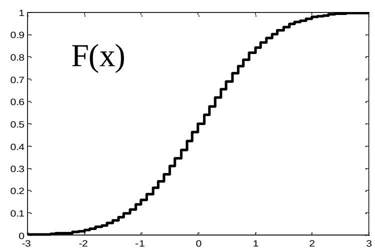
T co ham mat ño phan phoi: $f(t) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ (tra bang A).

Tích phan Laplace: $\varphi(x) = \int_0^x \dfrac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = P[0 \le T \le x], T \in N(0,1)$

That vay: $\varphi(x) = \int_0^x = \int_{-\infty}^x - \int_{-\infty}^0 = \Phi(x) - \Phi(0) = P[0 \le T \le x]$.

Ham φ(x) ñöôc tra ô bang B.

Ham phan phoi F(x) cua N(0,1)



Tính chat ham φ(x):

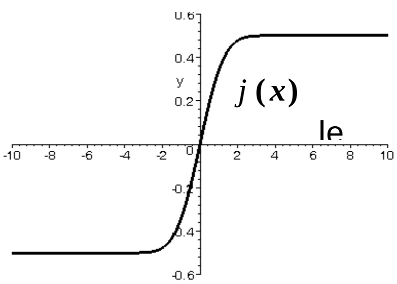
i/ $\varphi(-x) = -\varphi(x)$ le, va hien nhien ta co:

ii/ $\boxed{P[a \le T \le b] = \varphi(b) - \varphi(a)}$.

That vay:

$$P[a \le T \le b] = \int_a^b f(t)dt = \int_a^0 f(t)dt + \int_0^b f(t)dt =$$

$$= \int_0^b f(t)dt - \int_0^a f(t)dt = \varphi(b) - \varphi(a)$$



Ví du:

$T \in N(0,1), P(-3 < T < 1) = \varphi(1) + \varphi(3) = 0.3413 + 0.4987 = 0.84$

tra bang A: $f(0.25) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{0.25^2}{2}}$ = 0.3866.

Ta thöong dung bang F ñe tra ngöôc ham $\varphi(?)$: $\varphi(?) = 0.321$

Ta tính $\alpha = 1 - 2 \times 0.321$ =0.358, tra bang F: vôi α =0.358 cho ta x=0.919. Vay $\varphi(0.919) = 0.321$

6.6.2. Qui tac 3 sigma:

Bai toan: Cho $X \in N(\mu, \sigma^2)$, Hay giai phöông trình vôi an t_α :

$P\left[|X - \mu| \le t_\alpha . \sigma\right] = 1 - \alpha$? vôi $(1 - \alpha)$ cho tröôc va ñöôc goi la ño tin cay, va $\boxed{\varepsilon = t_\alpha . \sigma}$ goi la ño chính xac: Ta co:

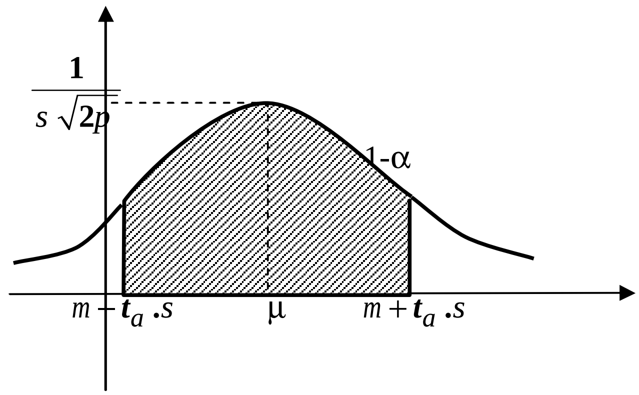
$$1 - \alpha = P\left[|X - \mu| \le t_\alpha . \sigma\right] = P\left[\frac{|X - \mu|}{\sigma} \le t_\alpha\right] = P\left[|T| \le t_\alpha\right] =$$

$= P[-t_\alpha \le T \le t_\alpha] = \varphi(t_\alpha) - \varphi(-t_\alpha) = 2\varphi(t_\alpha)$, tra baûng B tìm t_α.

$P[|X - \mu| \le t_\alpha.\sigma] = \boxed{1 - \alpha = 2\varphi(t_\alpha)}$, xaùc suaát naøy ñöôïc theå hieän treân hình veõ: Laø dieän tích phaàn gaïch cheùo.

Neáu laáy t_α =3, tra baûng B ta coù: $\varphi(t_\alpha)$=0.9973/ 2 = 0.4987 ==> $P[|X - \mu| \le 3\sigma] = 99.73\%$

Nghóa laø haàu heát (99.73%) giaù trò cuûa X sai leäch vôùi µ khoâng quaù 3 laàn σ. Töùc moät nhaø maùy saûn xuaát 10.000 saûn phaåm, cho pheùp coù 27 saûn phaåm hö, ñaây chæ phuø hôïp vôùi phaïm vi saûn xuaát nhoû, vaäy trong



phaïm vi saûn xuaát lôùn nhö moät tæ saûn phaåm thì cho pheùp 2.700.000 saûn phaåm hö !(keå caû hai ñaàu). Ngaøy nay caùc sieâu taäp ñoaøn saûn xuaát vôùi qui moâ ña quoác gia (coù haøng tæ saûn phaåm) ñaõ ñöa ra tieâu chuaån 6σ ñeå ño löôøng chaát löôïng saûn phaåm, nhaát laø tieâu chuaån ISO hieän nay ñeà cao vaø ñoøi hoûi tieâu chuaån 6σ moãi khi coâng ty muoán ñaêng caùi laáy chöùng nhaän ISO.

6.6.4. Caùch tính

Cho $X \in N(\mu, \sigma^2)$.Ñaët $T = \dfrac{X - \mu}{\sigma} \Rightarrow T \in N(0,1)$ (chöùng minh sau).

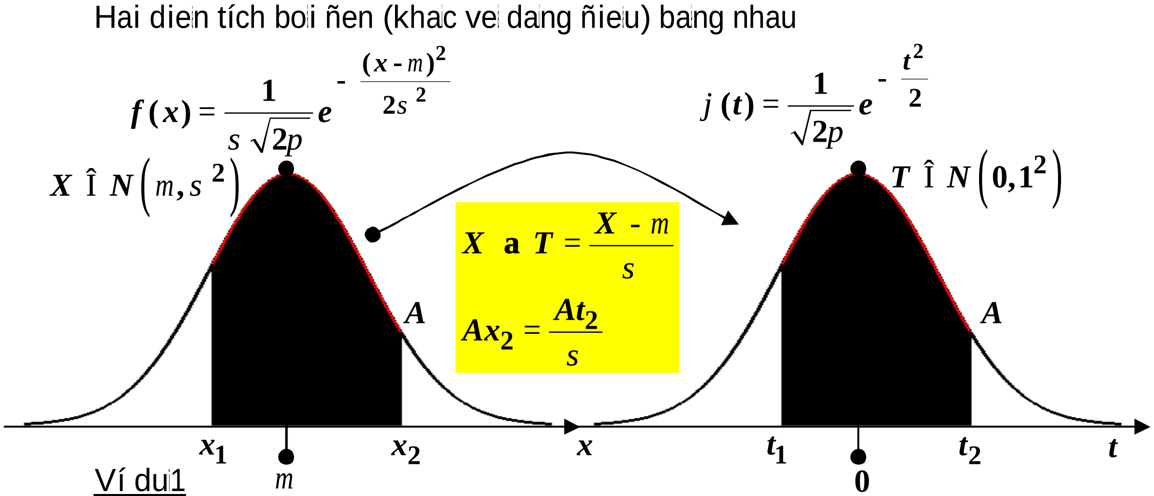
Phöông phaùp tính: Tính $P[x_1 \le X \le x_2]$.

Ñaët $t_1 = \dfrac{x_1 - \mu}{\sigma},\ t_2 = \dfrac{x_2 - \mu}{\sigma} \Rightarrow$

$P[x_1 \le X \le x_2] = P[x_1 \le \sigma T + \mu \le x_2] = P[t_1 \le T \le t_2] = \varphi(t_2) - \varphi(t_1)$.

$$\boxed{P[x_1 \le X \le x_2] = \varphi\left(\frac{x_2 - \mu}{\sigma}\right) - \varphi\left(\frac{x_1 - \mu}{\sigma}\right)}$$

Hai dieän tích bôûi ñen (khaùc veà daïng ñieäu) baèng nhau



Ví duï1

Troïng löôïng cuûa 1 loaïi saûn phaåm X coù phaân phoái chuaån vôùi $\mu = 10kg, \sigma = 0,5$. Tính tæ leä nhöõng saûn phaåm coù troïng löôïng töø $9,5kg \to 11kg$.

Giaûi

$$P[9,5 \le X \le 11] = \varphi\left(\frac{11-10}{0,5}\right) - \varphi\left(\frac{9,5-10}{0,5}\right) = \varphi(2) - \varphi(-1) = 0,82.$$

Excel: $NormDist(11,10,0.5,1) - NormDist(9.5,10,0.5,1)$ =0.82

Ví duï2

Thôøi gian X (tính baèng phuùt) cuûa moät khaùch haøng chôø xe buyùt laø bieán ngaãu nhieân coù phaân boá chuaån vôùi $\mu = 4.5, \sigma = 1.1$.

a/ Tính tæ leä khaùch haøng phaûi chôø xe buyùt töø $3,5 phut \to 6 phut$. Quaù 6 phuùt.

b/ Thôøi gian phaûi chôø toái thieåu laø bao nhieâu, neáu khoâng ñeå quaù 5% soá khaùch haøng phaûi chôø xe buyùt, maø vöôït quaù thôøi gian ñoù.

Giaûi

a/ $P[3,5 \le X \le 6] = \varphi\left(\frac{6-4.5}{1.1}\right) - \varphi\left(\frac{3,5-4,5}{1,1}\right) = \varphi(1.36) - \varphi(-0.91)$

$= \varphi(1.36) + \varphi(0.91) = 0.41309 + 0.31859 = 0.73168 = 73.17\%$.

Excel: $= NormDist(6,4.5,1.1,1) - NormDist(3.5,4.5,1.1,1)$ =73.17%

$P[6 < X] = \varphi\left(\frac{\infty - 4.5}{1.1}\right) - \varphi\left(\frac{6-4,5}{1,1}\right) = \frac{1}{2} - \varphi(1.36) = 0.5 - 0.41309 = 8.69\% =$ 1- $P[X < 6] =$ 1- $NormDist(6,4.5,1.1,1)$ =8.69%

b/ Goïi x_0 laø löôïng thôøi gian caàn tìm, ta coù:

$$P[x_0 < X] = \varphi(\infty) - \varphi\left(\frac{x_0 - 4,5}{1,1}\right) = \frac{1}{2} - \varphi\left(\frac{x_0 - 4,5}{1,1}\right) \le 0.05$$

$$\Rightarrow \varphi\left(\frac{x_0 - 4,5}{1,1}\right) \ge 0.45 = \varphi(1.65) \Rightarrow \frac{x_0 - 4,5}{1,1} \ge 1.65 \Rightarrow x_0 \ge 6.315.$$

Trong Excel: NORMSinv(0.5 +0.45) =1.65.

· Ñònh lyù giôùi haïn ñòa phöông (töøng ñieåm)

Vôùi p khoâng quaù gaàn 0 vaø 1 thì:

$$C_n^k p^k q^{n-k} \approx f(x=k) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi npq}} e^{-\frac{(k-np)^2}{2npq}} \quad (*)$$

Ñaët $\boxed{t = \frac{k-np}{\sqrt{npq}}}$ thì veá phaûi (*) laø $= \frac{1}{\sqrt{2\pi . npq}} e^{-\frac{t^2}{2}} = \boxed{\frac{1}{\sqrt{npq}} f(t)}$

vôùi $f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$, (tra baûng A), ñieàu naøy ñuùng vôùi moïi ñieåm $x = k$. Vaäy

$$\boxed{C_n^k p^k q^{n-k} \approx \frac{1}{\sqrt{npq}} f(t)}$$ (vôùi haøm f(t) tra baûng A)

3.2. Ñònh lyù giôùi haïn tích phaân (theo haøm phaân phoái)

Vôi p khong qua gan 0 va 1 thì:

Goi $X \in B(n,p)$ va $S_n = \dfrac{X - np}{\sqrt{npq}}$ thì $S_n \xrightarrow{F} N(0,1)$.

Noi cach khac $\boxed{X \in B(n,p) \xrightarrow[n \geq 30]{F} X \in N\left(m = np, s^2 = npq\right)}$

Vì hoi tu theo phan phoi (la tích phan tren khoang cua ham mat ño). Va vì the no con ñöôc goi la giôi han khoang.

Chöông 2: VECTÔ NGAU NHIEN (VTNN)

Trong thöc te, mot BNN chöa ñu bieu dien cho cuoc ñôi nhieu he luy, ta phai dung nhieu BNN hay con goi la vectô ngau nhien, töông tö trong toan cao cap, ño la ham nhieu bien.

Ñònh nghóa: Cap 2 ñai löôïng ngau nhien ñöôc xet ñong thôi (X,Y) ñöôc goi la 1 vector ngau nhien.

+ X, Y rôi rac $\Rightarrow (X,Y)$ rôi rac.

+ X, Y lien tuc $\Rightarrow (X,Y)$ lien tuc.

I. Luat phan phoi (X, Y rôi rac).

1.1. Bang phan phoi xac suat ñong thôi

Y \ X	y_1	y_2	...	y_j	...	y_n	P(X)
x_1	p_{11}	p_{12}	...	p_{1j}	...	p_{1n}	$p(x_1)$
x_2	p_{21}	p_{22}	...	p_{2j}	...	p_{2n}	$p(x_2)$
….	...						...
x_i	p_{i1}	p_{i2}	...	p_{ij}	...	p_{in}	$p(x_i)$
….	...						
x_m	p_{m1}	p_{m2}	...	p_{mj}	...	p_{mn}	$p(x_m)$
P(Y)	$p(y_1)$	$p(y_2)$	...	$p(y_j)$	...	$p(y_n)$	1

$P_{ij} = P[X = x_i, Y = y_j]$ (i = 1,...,m; j = 1,...,n) la xac suat ñe

$X = x_i$, $Y = y_j$ va $\sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} p\left(x_i, y_j\right) = 1$.

Ví du:

Y \ X	y_1	y_2	P(X)
x_1	0,05	0,30	0,35
x_2	0,45	0,20	0,65
P(Y)	0,50	0,50	1

1.2. Phan phoi xac suat le

• Phan phoi le cua X: (à cong theo chieu cua Y)

$\sum_{j=1}^{n} p\left(x_i, y_j\right) = \sum_{j=1}^{n} P[X = x_i, Y = y_j] = P[X = x_i] = p(x_i), \quad i = \overline{1,m}$ trong ño (cot xac suat cöc phai cua bang)

X	x_1 x_2 ... x_i ... x_m
P^X	$p(x_1)$ $p(x_2)$ ... $p(x_i)$ ... $p(x_m)$

Baing nay ñai ñöa ta vei mot BNN (cho dööi daing tan suat), mai chöông 2 ta ñai khaio sat.

Kyi voing lei cuia X: $E(X)=\sum_{i=1}^{m} x_i p(x_i)=\overline{x}$

Phöông sai lei cuia X: $s_x^2=\sum\left(x_i-\overline{x}\right)^2 p(x_i)=\frac{1}{m}\left(\sum_{1}^{m} x_i^2-m\overline{x}^2\right)$

• Phain phoi lei cuia Y: (à coing theo chieiu cuia X)

$\sum_{i=1}^{m} p(x_i,y_j)=\sum_{i=1}^{m} P[X=x_i,Y=y_j]=P[Y=y_j]=p(y_j),\quad j=\overline{1,n}$ trong ñoi (haing cuoii xaic suat cuia baing)

Y	y_1 y_2 ... y_j ... y_n
P^Y	$p(y_1)$ $p(y_2)$ ... $p(y_j)$ ... $p(y_n)$

Kyi voing lei cuia Y: $E(Y)=\sum_{j=1}^{n} y_j p(y_j)=\overline{y}$

Phöông sai lei cuia Y: $s_y^2=\sum\left(y_j-\overline{y}\right)^2 p(y_j)=\frac{1}{n}\left(\sum_{1}^{n} y_i^2-n\overline{y}^2\right)$

1.3. Phain phoi xaic suat coi ñieiu kiein.

$$p\left(x_i/y_j\right)=P[X=x_i/Y=y_j]=\frac{P[X=x_i,Y=y_j]}{P[Y=y_j]}=\frac{P[x_i,y_j]}{p(y_j)},\quad i=\overline{1,m}$$

$$p\left(y_j/x_i\right)=P[Y=y_j/X=x_i]=\frac{P[X=x_i,Y=y_j]}{P[X=x_i]}=\frac{P[x_i,y_j]}{p(x_i)},\quad j=\overline{1,n}$$

Chui yi: Do ñieiu kiein ñay ñui cuia xaic suat, nein

$$\sum_{i=1}^{m} p\left(x_i/y_j\right)=1\Leftrightarrow\sum_{x} p\left(x/y\right)=1$$

$$\sum_{j=1}^{n} p\left(y_j/x_i\right)=1\Leftrightarrow\sum_{y} p\left(y/x\right)=1$$

<u>Ví dui 3:</u>

Moit hoip coi 2 sain phaim xau vai 3 sain phaim (sp) toit. Lay 2 lain, moii lain 1 sain phaim.

Ñait X = $\begin{cases}1, \text{neiu lain 1 lay ñöôic sp tot}\\ 0, \text{neiu lain 1 lay ñöôic sp xaiu}\end{cases}$

Y = $\begin{cases}1, \text{neiu lain 2 lay ñöôic sp toit}\\ 0, \text{neiu lain 2 lay ñöôic sp xaiu}\end{cases}$.

a/ Tìm phan phoii ñoing thôii cuia (X, Y).

b/ Tìm phan phoii lei cuia X, Y.

<u>Giai</u>

Coi 4 tröông hôip:

P[X=0, Y=0] = P[X=0].P[Y=0/X=0] = $\frac{2}{5}.\frac{1}{4}=0,1$

P[X=0, Y=1] = P[X=0].P[Y=1/X=0] = $\frac{2}{5}.\frac{3}{4}=0,3$

P[X=1, Y=0] = P[X=1].P[Y=0/X=1] = $\frac{3}{5}.\frac{2}{4}=0,3$

P[X=1, Y=1] = P[X=1].P[Y=1/X=1] = $\frac{3}{5}.\frac{2}{4}=0,3$.

X \ Y	0	1	P(X)
0	0,1	0,3	0,4
1	0,3	0,3	0,6
P(Y)	0,4	0,6	1

Ví du1: Thong ke ve thu nhap theo giôi tính cua cong nhan det may cua khu cong nghiep cho trong bang sau:

Y: Thu nhap Loai 1= 1trieu, Loai 2= 2trieu, Loai 3= 3trieu, X: giôi tính:0=nö, 1=nam)

X \ Y	1	2	3	P(X)
0	0.1	0.25	0.16	0.51
1	0.15	0.22	0.12	0.49
P(Y)	0.25	0.47	0.28	1

a/ Lap bang phan phoi xac suat cua thu nhap, cua giôi tính.

b/ Thu nhap co ñoc lap vôi giôi tính khong?

c/ Tìm xac suat lay NN mot ngöôi co thu nhap tö loai 2 trô len.

d/ Lap bang phan phoi xac suat cua thu nhap cua nam, tính trung bình thu nhap cua nam.

Giai

a/ Bang phan phoi xac suat cua thu nhap:

Y	1	2	3
P(Y)	0.25	0.47	0.28

Bang phan phoi xac suat cua giôi tính:

X	0	1
P(X)	0.51	0.49

b/ $P\left(X=0,Y=1\right)=0.1\neq P\left(X=0\right)\times P\left(Y=1\right)=0.51\times 0.25$

nen giöa 2 BNN thu nhap khong ñoc lap vôi BNN giôi tính.

c/ P= 0.47 + 0.28 = 0.75

d/

Y	1	2	3
$P\left(Y/X=1\right)$	$\frac{0.15}{0.49}=0.306$	$\frac{0.22}{0.49}=0.449$	$\frac{0.12}{0.49}=0.244$

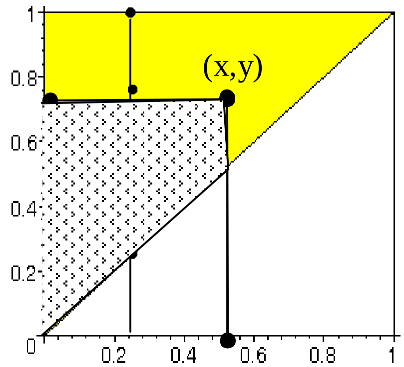
$E\left(Y/X=1\right)=1\times 0.306+2\times 0.449+3\times 0.2449=1.938$

trung bình thu nhap cua cong nhan nam la 1.938 trieu ñong. ¢

Ví du 3:

Cho ham mat ño ñong thôi cua BNN X, Y

nhö sau $f(x,y)=\begin{cases} kxy^2, & 0<x<y<1 \\ 0 \end{cases}$



a/ Tìm k?

b/ Tìm $f_X(x)$, $f_Y(y)$, tính E(Y).

$$f_Y(y|\ x),\ f_X(x|\ y),\ f\left({(x,y)}\Big/{x>\frac{1}{4}} \right)$$

c/ Tính $P\left(\frac{1}{2}<Y \middle|\ X=\frac{1}{4}\right)$.

d/ Tính $P\left(Y<\frac{3}{4}\middle|\ X=\frac{1}{4}\right)$, $P\left(Y<\frac{3}{4}\middle|\ X>\frac{1}{4}\right)$, $P\left(Y<\frac{3}{4},\ X>\frac{1}{4}\right)$.

e/ Tìm ham hoi qui cua Y theo X.

f/ Tìm ham hoi qui cua X theo Y.

g/ Tìm ky vong X,Y.

h/ Tìm ky vong E(XY).

i/ Tìm phöông sai X,Y.

j/ Tìm ham phan phoi ñong thôi F(x,y).

k/ Tìm ham $\mathrm{var}\left({Y}/{X}\right)$.

l/ Lap bang phan phoi U=X+Y, V=X-Y, lap $f_U(u)$, $f_V(v)$

m/ Neu goi $Z=q\,X+(1-q)\,Y$ la löông tien ñau tö cho ca hai chæ tieu X va Y thong qua tham so $q\in[0,1]$, hay tìm tham so q ñe sö ñau tö nay la ít rui ro nhat.

<u>Giai:</u>

Khi lay tích phan 2 lôp, tuy ta co the lay thö tö can tích phan hôp ly.

a/ $$1=\int_{y=0}^{y=1}\int_{x=0}^{x=y}...=k\int_{x=0}^{x=1}\int_{y=x}^{y=1}xy^2dxdy=k\int_{x=0}^{x=1}\frac{x\left(1-x^3\right)}{3}dx=\frac{k}{10}\Rightarrow k=10$$

b/ $$\begin{cases} 0<x<1 \\ f_X(x)=\displaystyle\int_{y=x}^{y=1}f(x,y)dy=10\int_{y=x}^{y=1}xy^2dy=10\frac{xy^3}{3}\Bigg|_{y=x}^{y=1}=\frac{10x\left(1-x^3\right)}{3} \end{cases}$$

$$\begin{cases} 0<y<1 \\ f_Y(y)=\displaystyle\int_{x=0}^{x=y}f(x,y)dx=10\int_{x=0}^{x=y}xy^2dx=5x^2y^2\Bigg|_{x=0}^{x=y}=5y^4 \end{cases}$$

$E(Y)=\displaystyle\int_{y=0}^{y=1}y.f_Y(y)dy=\int_{y=0}^{y=1}y.5y^4dy=\frac{5}{6}$, hay

$$E(Y)=\iint y.f(x,y)dxdy=10\int_{y=0}^{y=1}\left(\int_{x=0}^{x=y}y.xy^2dx\right)dy=\int_{y=0}^{y=1}5y^5dy=\frac{5}{6}$$

hay $E(Y)=10\int_{x=0}^{x=1}\left(\int_{y=x}^{y=1} y.xy^2dy\right)dx=\frac{5}{2}\int_{x=0}^{x=1}x\left(1-x^4\right)dx=\frac{5}{6}$,

$$f_Y(y|\ x)=\frac{f(x,y)}{f_X(x)}=\frac{10xy^2}{\frac{10}{3}x\left(1-x^3\right)}=\frac{3y^2}{\left(1-x^3\right)},\quad 0<x<y<1$$

$$f_X(x|\ y)=\frac{f(x,y)}{f_Y(y)}=\frac{10xy^2}{5y^4}=\frac{2x}{y^2},\quad 0<x<y<1$$

Ta coù: $P\left(X>\frac{1}{4}\right)=\int_{x=\frac{1}{4}}^{x=1}10xdx\int_{y=x}^{y=1}y^2dy=\frac{459}{512}$.

Vaäy $f\left((x,y)\Big/ x>\frac{1}{4}\right)=\frac{f(x,y)}{P\left(X>\frac{1}{4}\right)}=\frac{5120xy^2}{459}$

c/ $P\left(\frac{1}{2}<Y\Big|\ X=\frac{1}{4}\right)=\int_{y=\frac{1}{2}}^{y=1}f_Y(y|\ x)dy=\int_{\frac{1}{2}}^{1}\frac{3y^2}{\left(1-0.25^3\right)}dy=\frac{8}{9}$.

d/ $P\left(Y<\frac{3}{4}\Big|\ X=\frac{1}{4}\right)=\int_{y=\frac{1}{4}}^{y=\frac{3}{4}}f_Y(y|\ x)dy=\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{3y^2}{\left(1-0.25^3\right)}dy=\frac{26}{63}$.

$$P\left(Y<\frac{3}{4}\Big|\ X>\frac{1}{4}\right)=\iint_{Y<\frac{3}{4},X>\frac{1}{4}}f\left((x,y)\Big/ x>\frac{1}{4}\right)dxdy=$$

$$=\iint_{Y<\frac{3}{4},X>\frac{1}{4}}\frac{5120xy^2}{459}dxdy=\frac{5120}{459}\int_{x=\frac{1}{4}}^{x=\frac{3}{4}}xdx\int_{y=x}^{y=\frac{3}{4}}y^2dy=\frac{298}{1377}=\frac{149}{768}.\frac{5120}{459}$$

So saùnh vôùi: $P\left(Y<\frac{3}{4},\ X>\frac{1}{4}\right)=\int_{x=\frac{1}{4}}^{x=\frac{3}{4}}10xdx\int_{y=x}^{y=\frac{3}{4}}y^2dy=\frac{149}{768}$

e/ $E(Y/x)=\int_{y=x}^{y=1}yf_Y\left(y\big/ x\right)dy=\Psi(x)=\int_{y=x}^{y=1}y\frac{3y^2}{\left(1-x^3\right)}dy=\frac{3\left(1-x^4\right)}{4\left(1-x^3\right)}$

f/ $E(X/y)=\int_{x=0}^{x=y}xf_X\left(x\big/ y\right)dx=\Psi(y)=\int_{x=0}^{x=y}x\frac{2x}{y^2}dx=\frac{2y}{3}$

Haøm phaân phoái: $F(x,y)=\int_{u=0}^{u=x}du\int_{v=u}^{v=y}f(u,v)dv=\frac{-2}{3}x^5+\frac{5}{3}y^3x^2$

ta co the thöi lai cac bieu thöc: $\dfrac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$,

$$\frac{\partial F(x,y)}{\partial x} \neq f_X(x),\ \frac{\partial F(x,y)}{\partial y} \neq f_Y(y)$$

g/ $E(Y) = 10 \int\limits_{x=0}^{x=1} \int\limits_{y=x}^{y=1} y \times xy^2 dxdy = \dfrac{5}{6}$, hay co the

$$E(Y) = \int\limits_{y=0}^{y=1} y \times f_Y(y)dy = \int\limits_{y=0}^{y=1} y \times 5y^4 dy = \frac{5}{6}$$

k/ $\operatorname{var}\left(Y/X\right) = E\left(y^2/x\right) - E^2\left(y/x\right)$

$$E\left(y^2/x\right) = \int\limits_{y=x}^{y=1} y^2 f\left(y/x\right) dy = \int\limits_{y=x}^{y=1} y^2 \frac{3y^2}{\left(1-x^3\right)} dy = \frac{3\left(1-x^5\right)}{5\left(1-x^3\right)}$$

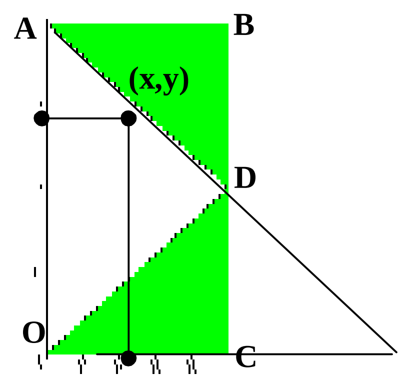
$$\operatorname{var}\left(Y/X\right) = \frac{3\left(1-x^5\right)}{5\left(1-x^3\right)} - \left[\frac{3\left(1-x^4\right)}{4\left(1-x^3\right)}\right]^2$$

Cac cau con lai sinh vien töi lam ¢

Ví du 4:

Cho hm mật ộ ồng thời của X v Y

$$f(x,y) = \begin{cases} k\left(x^2 + \dfrac{xy}{2}\right), & 0 < x < 1,\ 0 < y \le 2 \\ 0, & \text{noi khac} \end{cases}$$



a) Tìm k.
b) Tìm mật ộ phn phối lề của X v của Y.
c) Tính kỳ vọng E(X), E(Y).
d) Tính xc suất $P(X > Y)$.
e) Tính xc suất $P(X + Y > 2)$.
f) Tìm ham hoi qui cua Y theo X.
g) Tìm ham hoi qui cua X theo Y.
h/ Tìm ham phan phoi ñong thôi F(x,y).

Giai:

a/ $k\int\limits_0^2\int\limits_0^1\left(x^2 + \dfrac{xy}{2}\right)dxdy = k\int\limits_0^2\left(\dfrac{1}{3} + \dfrac{y}{4}\right)dy = \dfrac{7k}{6} = 1 \Rightarrow k = \dfrac{6}{7}$

(mien lay tích phan la hình chöi nhat OABC, xem hình ve)

b/ $\begin{cases} 0 < x < 1 \\ f_X(x) = \displaystyle\int\limits_{y=0}^{y=2} f(x,y)dy = \frac{6}{7}\int\limits_{y=0}^{y=2}\left(x^2 + \frac{xy}{2}\right)dy = \frac{6x}{7}(1+2x) \end{cases}$

$$\begin{cases} 0 < y < 2 \\ f_Y(y) = \int\limits_{x=0}^{x=1} f(x,y)dx = \frac{6}{7}\int\limits_{x=0}^{x=1}\left(x^2 + \frac{xy}{2}\right)dx = \frac{2}{7} + \frac{6y}{14} \end{cases}$$

c) kỳ vọng của X: $E(X) = \int\limits_{-\infty}^{+\infty} x f_X(x)dx$.

d) $P(X > Y) = \iint\limits_{x>y} f(x,y)dxdy = k\int\limits_0^1 dx\int\limits_0^x\left(x^2 + \frac{xy}{2}\right)dy$

(mien lay tích phan la mien tam giac döôi ODC, xem hình ve)

e) $P(X + Y > 2) = \iint\limits_{x+y>2} f(x,y)dxdy = k\int\limits_0^1 dx\int\limits_{y=-x+2}^{y=2}\left(x^2 + \frac{xy}{2}\right)dy$

(mien lay tích phan la mien tam giac tren ABD, xem hình ve)

Ham phan phoi: $F(x,y) = \frac{2}{7}yx^3 + \frac{3}{28}y^2x^2$

Cac cau con lai xem nhö bai tap ¢

III. Hiep phöông sai, he so töông quan: $\mathbf{cov}(X,Y), s(X,Y)$

Neu X, Y ñoc lap thì ==> $E(XY) = E(X).E(Y)$ (chieu ngöôc lai khong ñung, xem ví du döôi ñay), luc ño hieu cua chung bang khong, trong tröông hôp hieu cua chung khac zero thì X, Y khong ñoc lap, ta goi hieu ño la hiep phöông sai, nhöng tröôc het ta ñònh nghóa nhö sau:

$$\mathbf{cov}(X,Y) = s(X,Y) = E\left[(X - E(X))\times(Y - E(Y))\right]$$

(co-oporation-variance), ñöôc goi la hiep phöông sai.

• He so töông quan cua hai BNN X va Y la so:

$$\boxed{r(X,Y) = \frac{\mathbf{cov}(X,Y)}{\sqrt{\mathbf{var}(X).\mathbf{var}(Y)}} = \frac{\mathbf{cov}(X,Y)}{s(X).s(Y)} = \frac{s(X,Y)}{s_X.s_Y}}$$

3.1. Ñònh ly ve cac tính chat ñac tröng cua hai BNN X va Y:

1/ $\mathbf{cov}(X,Y) = \mathbf{cov}(Y,X) = E(XY) - E(X).E(Y) \neq 0$

Töông töï $\mathbf{var}(X) = E(X - EX)^2 = E(X^2) - [E(X)]^2 \geq 0$ ta co

$\mathbf{var}(X,Y) = s(X,Y) = E(XY) - E(X).E(Y) = \mathbf{cov}(X,Y)$.

Nhöng khong co $\mathbf{var}(X,Y) \geq 0$ vôi $\forall X,Y$, no co the am.

Neu cho döôi dang tan suat cac cap (X,Y) $= (x_i, y_i), i = \overline{1,n}$, thì ñònh nghóa cov(X,Y) con the hien qua cong thöc:

$$\mathbf{cov}(X,Y) = \frac{1}{n}\sum_{i=1}^{i=n}\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right) = \frac{\sum\limits_{i=1}^{i=n} x_i y_i}{n} - \frac{\sum\limits_{i=1}^{i=n} x_i}{n} \times \frac{\sum\limits_{i=1}^{i=n} y_i}{n} =$$

$$= \overline{XY} - \overline{X}.\overline{Y} = \boxed{\frac{1}{n}\left(\sum_{i=1}^{i=n} x_i y_i - n\bar{x}.\bar{y}\right) = \mathbf{cov}(X,Y)}$$

Khi $Y \equiv X$ thì $\mathbf{cov}(X,X) = \mathbf{var}(X) = s^2(X) = E(X^2) - [E(X)]^2$

2/ $\operatorname{cov}(X+X',Y)=\operatorname{cov}(X,Y)+\operatorname{cov}(X',Y)$

3/ $\operatorname{cov}(kX,Y)=k\operatorname{cov}(X,Y)$

4/ $\operatorname{var}(X+Y)=\operatorname{var}(X)+\operatorname{var}(Y)+2\operatorname{cov}(X,Y)$.

Suy ra toång quaùt nhö sau: Nếu X, Y là <u>không độc lập</u> thì:

5/ $\boxed{\operatorname{var}(aX+bY)=a^2\operatorname{var}(X)+b^2\operatorname{var}(Y)+2ab\operatorname{cov}(X,Y)}$

Chuù yù: $\operatorname{var}(X+X)=2\operatorname{var}(X)+2\times1\times1\operatorname{cov}(X,X)=4\operatorname{var}(X)$

Hay goïn hôn: $\operatorname{var}(X+X)=\operatorname{var}(2X)=2^2\operatorname{var}(X)=4\operatorname{var}(X)$

6/ $[\operatorname{cov}(X,Y)]^2\le\operatorname{var}(X).\operatorname{var}(Y)\Rightarrow|r|\le1$

7/ Neáu X vaø Y ñoäc laäp ta coù cov(X,Y) =0 suy ra r=0.

8/ Neáu r = ±1 khi vaø chæ khi toàn taïi a $\neq 0$, d $\in$ R: X= d+aY.

Luùc naøy ta noùi X vaø Y töông quan tuyeán tính hoaøn toaøn.

Trong tröôøng hôïp r=0 ta noùi X vaø Y khoâng töông quan. Vaäy neáu X vaø Y ñoäc laäp nhau thì chuùng khoâng töông quan. Tieác raèng ñieàu ngöôïc laïi khoâng ñuùng neân khoâng theå duøng heä soá töông quan r ñeå bieåu dieãn tính ñoäc laäp cuûa hai BNN X vaø Y.

9/ $\operatorname{var}\left(\dfrac{X}{\sigma(X)}\pm\dfrac{Y}{\sigma(Y)}\right)=2(1\pm r(X,Y))\ge0$

<u>Ví duï2:</u>

X (đơn vị: kg) và Y (đơn vị: cm) là hai chỉ tiêu của một loại sản phẩm. Điều tra một mẫu ta có bảng số liệu <u>tần số</u> sau

Y \ X	0	1	2	3
0	1			
1		2	3	
2		3	2	1

a/ Tính phaân phoái leà cuûa X, Y.

b/ Tính kyø voïng cuûa X, Y, phöông sai cuûa X, Y.

c/ Tính cov(X, Y), r_{YX}, $\operatorname{cov}(X,Y^2)$

d/ Tìm <u>haøm hoài qui lyù thuyeát rôøi raïc</u> E(Y/ X).
nghieäm laïi: $E(Y)=E_X\left[E\left(Y/X\right)\right]$, tính $\operatorname{var}\left(Y/X\right)$

e/ Tìm phaân phoái coù ñieàu kieän (Y/ X=0), (Y/ X=1), (Y/ X=2), (Y/ X=3).

f/ Tìm phaân phoái coù ñieàu kieän (X/ Y=0), (X/ Y =1), (X/ Y =2).

g/ Tìm <u>hàm hồi quy mẫu</u> của Y theo X. (chương 8)

h/ Laäp baûng phaân phoái T=X+Y, tính kyø voïng vaø phöông sai cuûa T.

i/ Nếu gọi $Z=qX+(1-q)Y$ laø löôïng tieàn ñaàu tö cho caû hai chæ tieâu X vaø Y thoâng qua tham soá $q\in[0,1]$, haõy tìm tham soá q ñeå söï ñaàu tö naøy laø ít ruûi ro nhaát.

j/ Tìm $P(Y<2)$, $P(Y<2,X>0)$, $P(Y=1,X\ge1)$

Giai:

Y \ X	0	1	2	3	P(Y)
0	$\frac{1}{12}$				$\frac{1}{12}$
1		$\frac{2}{12}$	$\frac{3}{12}$		$\frac{5}{12}$
2		$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{6}{12}$
P(X)	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	1

Ta chuyển về bảng tần suất như trn :

Bam may 570MS: Mode --> Reg -->Lin, vao 6 o dö lieu:

$\mathbf{0,0;\ 1}\ \boxed{M+}\quad \mathbf{1,1;\ 2}\ \boxed{M+}\quad \mathbf{1,2;\ 3}\ \boxed{M+}$

$\mathbf{2,1;\ 3}\ \boxed{M+}\quad \mathbf{2,2;\ 2}\ \boxed{M+}\quad \mathbf{3,2;\ 1}\ \boxed{M+}$

Bam: $\uparrow$ ***Sum*** va $\uparrow$ ***S* var**, ta co cac ket qua sau:

$$\sum x^2 = 34,\ \sum x = 18,\ \sum y^2 = 29,\ \sum y = 17,\ \sum xy = 28,\ n = 12$$

Ví du: $\sum y^2 = 0^2 \times 1 + 1^2 \times (2+3) + 2^2 \times (3+2+1) = 29$

$$\sum x^2 = 0^2 \times 1 + 1^2 \times (2+3) + 2^2 \times (3+2) + 3^2 \times 1 = 34 \text{.........}$$

$$\sum xy = \left(0\times 0\times \boxed{1}\right) + \left(1\times 1\times \boxed{2}\right) + \left(2\times 1\times \boxed{3}\right) +$$

$$+\left(1\times 2\times \boxed{3}\right) + \left(2\times 2\times \boxed{2}\right) + \left(3\times 2\times \boxed{1}\right) = 28.$$

$\overline{X} = \dfrac{\sum x = 18}{12}$ =1.5, s_X =0.76376, vôi phöông sai ta dung cong thöc nhö chöông 2:

$$s_X^2 = \frac{1}{n}\sum\left(X_i - \overline{X}\right)^2 = \frac{1}{n}\left(\sum X_i^2 - n\overline{X}^2\right)$$

$\overline{Y} = \dfrac{\sum y = 17}{12}$ = 1.41666, s_Y =0.6400, r= r_{YX} =0.42614,

Ham hoi qui mau: y=0.35714x + 0.880952

a/ Tính phan phoi le cua X, Y: nhö tren bang.

b/ Tính ky vong, phöông sai cua X, Y.

$$E(Y) = \sum_{y=0}^{y=2}\sum_{x=0}^{x=3} yf(x,y) = \sum_{y=0}^{y=2} yf_Y(y) =$$

$=\overline{Y} = E(Y) = \dfrac{\sum y}{12} = \dfrac{17}{12}$ = 1.41666. Co the tính lai:

$$\overline{Y} = E(Y) = 0\times\frac{1}{12} + 1\times\frac{5}{12} + 2\times\frac{6}{12} = \frac{17}{12} = 1.41666$$

$$s_Y^2 = \frac{1}{n}\left(\sum Y_i^2 - n\overline{Y}^2\right) = \frac{1}{12}\left(29 - 12\times\left(\frac{17}{12}\right)^2\right) = \frac{59}{144} = 0.4097222$$

==> $s_Y = \sqrt{\text{var}(Y)} = 0.6400$. Coù theå tính laïi: $s_Y^2 = E(Y^2) - EY^2$

$$\text{var}(Y) = \left(0^2\times\frac{1}{12} + 1^2\times\frac{5}{12} + 2^2\times\frac{6}{12}\right) - \left(\frac{17}{12}\right)^2 = \frac{29}{12} - \left(\frac{17}{12}\right)^2 = \frac{59}{144}$$

$$E(X) = \sum_{x=0}^{x=3}\sum_{y=0}^{y=2} xf(x,y) = \sum_{x=0}^{x=3} xf_X(x) = \overline{X} = E(X) = \frac{\sum x}{12} = \frac{18}{12} = 1.5$$

Coù theå tính laïi:

$$\overline{X} = E(X) = 0\times\frac{1}{12} + 1\times\frac{5}{12} + 2\times\frac{5}{12} + 3\times\frac{1}{12} = \frac{18}{12} = 1.5$$

$$s_X^2 = \frac{1}{n}\left(\sum X_i^2 - n\overline{X}^2\right) = \frac{1}{12}\left(34 - 12\times\left(\frac{18}{12}\right)^2\right) = \frac{84}{144} = 0.583333$$

==> $s_X = \sqrt{\text{var}(X)} = 0.76376$. Coù theå tính laïi: $s_X^2 = E(X^2) - EX^2$

$$\text{var}(X) = \left(0^2\times\frac{1}{12} + 1^2\times\frac{5}{12} + 2^2\times\frac{5}{12} + 3^2\times\frac{1}{12}\right) - \left(\frac{18}{12}\right)^2 = \frac{34}{12} - \left(\frac{18}{12}\right)^2 = \frac{84}{144}$$

c/ Tính cov(X, Y), r_{YX} .

$$E(XY) = \sum_{x=0}^{x=3}\sum_{y=0}^{y=2} xyf(x,y) = \overline{XY} = \frac{\sum XY}{n} = \frac{28}{12} = 2.33333.$$

Coù theå tính laïi: $E(XY) = \left(0\times0\times\boxed{1}\times\frac{1}{12}\right) + \left(1\times1\times\boxed{2}\times\frac{1}{12}\right) + \left(2\times1\times\boxed{3}\times\frac{1}{12}\right) +$

$$+\left(1\times2\times\boxed{3}\times\frac{1}{12}\right) + \left(2\times2\times\boxed{2}\times\frac{1}{12}\right) + \left(3\times2\times\boxed{1}\times\frac{1}{12}\right) = 28\times\frac{1}{12} = 2.33333.$$

$$\text{cov}(X,Y) = E(XY) - E(X)\times E(Y) = \frac{28}{12} - \frac{18}{12}\times\frac{17}{12} = \frac{30}{144} = \frac{5}{24} = 0.20833$$

$$r_{YX} = \frac{\text{cov}(X,Y)}{s_X s_Y} = \frac{\frac{30}{144}}{0.76376\times0.64} = 0.426208$$

Ñeå tính $\text{cov}(X,Y^2) = 0.3751$

Baám maùy 570MS: Mode --> Reg -->Lin, vaøo 6 oâ döõ lieäu:

$0,0^2;\ 1\ \boxed{M+}$, $1,1^2;\ 2\ \boxed{M+}$, $1,2^2;\ 3\ \boxed{M+}$

$2,1^2;\ 3\ \boxed{M+}$, $2,2^2;\ 2\ \boxed{M+}$, $3,2^2;\ 1\ \boxed{M+}$

$$\text{cov}(X,Y^2) = E(XY^2) - E(X)\times E(Y^2) = \frac{1}{n}\left(\sum xy^2 - n\overline{x}.\overline{y^2}\right) = \frac{3}{8}.$$

Vôùi $\overline{XY^2} = \frac{1}{12}\left[\begin{array}{l}1\times0\times0^2 + 2\times1\times1^2 + 3\times2\times1^2 + 3\times1\times2^2 + \\ +2\times2\times2^2 + 1\times3\times2^2\end{array}\right] = \frac{48}{12}$

d/ Tìm haøm hoài qui lyù thuyeát rôøi raïc E(Y/ X), taïi 4 ñieåm

X=0, X=1, X=2, X=3.

$$E\left(Y/X\right)=\sum_y yP\left(y/x\right)=\sum_y y\frac{P(x,y)}{P(x)}=\frac{1}{P(x)}\sum_y yP(x,y)$$

Tai X=0:

$$E\left(Y/X=0\right)=\frac{1}{P(x)}\sum_y yP(x,y)=\frac{1}{\frac{1}{12}}\left[0\times\frac{1}{12}+1\times\frac{0}{12}+2\times\frac{0}{12}\right]=0$$

Tai X=1:

$$E\left(Y/X=1\right)=\frac{1}{P(x)}\sum_y yP(x,y)=\frac{1}{\frac{5}{12}}\left[0\times\frac{0}{12}+1\times\frac{2}{12}+2\times\frac{3}{12}\right]=\frac{8}{5}=1.6$$

Tai X=2:

$$E\left(Y/X=2\right)=\frac{1}{P(x)}\sum_y yP(x,y)=\frac{1}{\frac{5}{12}}\left[0\times\frac{0}{12}+1\times\frac{3}{12}+2\times\frac{2}{12}\right]=\frac{7}{5}=1.4$$

Tai X=3:

$$E\left(Y/X=3\right)=\frac{1}{P(x)}\sum_y yP(x,y)=\frac{1}{\frac{1}{12}}\left[0\times\frac{0}{12}+1\times\frac{0}{12}+2\times\frac{1}{12}\right]=2$$

Vay ta co bang tính toan:

X	0	1	2	3
$E\left(Y/X\right)$	0	$\frac{8}{5}$	$\frac{7}{5}$	2
$P(X)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$
$E\left(Y^2/X\right)$	0	$\frac{14}{5}$	$\frac{11}{5}$	4
$\text{var}\left(Y/X\right)$	0	$\frac{6}{25}$	$\frac{6}{25}$	0

nghiem lai: $E(Y)=E_X\left[E\left(Y/X\right)\right]$

==> $E_X\left[E\left(Y/X\right)\right]=0\times\frac{1}{12}+\frac{8}{5}\times\frac{5}{12}+\frac{7}{5}\times\frac{5}{12}+2\times\frac{1}{12}=\frac{17}{12}=E(Y)$

Ñe tính $\text{var}\left(Y/X\right)$, ta phai tính phan phoi $\left(Y^2/X=0,1,2\right)$, tai cac ñiem $X=0,1,2$, gia sö nhö tai X=1:

$\left(Y^2/X=1\right)$	0	1^2	2^2	
$P\left(Y^2/X=1\right)$	0	$\frac{2}{5}$	$\frac{3}{5}$	$\Rightarrow E\left(Y^2/X=1\right)=\frac{14}{5}$

$\mathbf{var}\left(Y/X\right)=E\left(y^2/x\right)-E^2\left(y/x\right)$, tính theo töng ñiem nhö bang tren. Ví du:

$$\mathbf{var}\left(Y/X=1\right)=E\left(y^2/x=1\right)-E^2\left(y/x=1\right)=\frac{14}{5}-\left(\frac{8}{5}\right)^2=\frac{6}{25},$$

töông töi cac ñiem khac.

e/ Phan phoi co ñieu kien (Y/ X=0), (Y/ X=1), (Y/ X=2), (Y/X=3).

Tai X=0, $P\left(Y=0/X=0\right)=\dfrac{P(X=0\cap Y=0)}{P(X=0)}=\dfrac{\frac{1}{12}}{\frac{1}{12}}=1$.

$\left(Y/X=0\right)$	0	1	2	
$P\left(Y/X=0\right)$	1	0	0	$\Rightarrow E\left(Y/X=0\right)=0$

Tai X=1, $P\left(Y=1/X=1\right)=\dfrac{P(X=1\cap Y=1)}{P(X=1)}=\dfrac{\frac{2}{12}}{\frac{5}{12}}=\dfrac{2}{5}$,

$$P\left(Y=2/X=1\right)=\dfrac{P(X=1\cap Y=2)}{P(X=1)}=\dfrac{\frac{3}{12}}{\frac{5}{12}}=\dfrac{3}{5}$$

$\left(Y/X=1\right)$	0	1	2	
$P\left(Y/X=1\right)$	0	$\frac{2}{5}$	$\frac{3}{5}$	$\Rightarrow E\left(Y/X=1\right)=\frac{2}{5}+\frac{6}{5}=\frac{8}{5}$

Tai X=2.

$\left(Y/X=2\right)$	0	1	2	
$P\left(Y/X=2\right)$	0	$\frac{3}{5}$	$\frac{2}{5}$	$\Rightarrow E\left(Y/X=2\right)=\frac{3}{5}+\frac{4}{5}=\frac{7}{5}$

Tai X=3.

$\left(Y/X=3\right)$	0	1	2	
$P\left(Y/X=3\right)$	0	0	1	$\Rightarrow E\left(Y/X=3\right)=0+2=2$

ta co the so sanh ket qua $E\left(Y/X_i\right)$ vôi ket qua cau tren, nhö nhau.

f/ Tìm phan phoi co ñieu kien (X/ Y=0), (X/ Y =1), (X/ Y =2).

Tai Y=0, $P\left(X=0/Y=0\right)=\dfrac{P(X=0\cap Y=0)}{P(Y=0)}=\dfrac{\frac{1}{12}}{\frac{1}{12}}=1$.

$\left(X/Y=0\right)$	0	1	2	3	
$P\left(X/Y=0\right)$	1	0	0	0	$\Rightarrow E\left(X/Y=0\right)=0$

Tai Y=1, $P\left(X=1/_{Y=1}\right)=\dfrac{P(X=1\cap Y=1)}{P(Y=1)}=\dfrac{\frac{2}{12}}{\frac{5}{12}}=\dfrac{2}{5}$,

$$P\left(X=2/_{Y=1}\right)=\dfrac{P(X=2\cap Y=1)}{P(Y=1)}=\dfrac{\frac{3}{12}}{\frac{5}{12}}=\dfrac{3}{5}$$

$\left(X/_{Y=1}\right)$	0	1	2	3	
$P\left(X/_{Y=1}\right)$	0	$\frac{2}{5}$	$\frac{3}{5}$	0	$\Rightarrow E\left(X/_{Y=1}\right)=\frac{2}{5}+\frac{6}{5}=\frac{8}{5}$

Tai Y=2, $P\left(X=1/_{Y=2}\right)=\dfrac{P(X=1\cap Y=2)}{P(Y=2)}=\dfrac{\frac{3}{12}}{\frac{6}{12}}=\dfrac{3}{6}$,

$$P\left(X=2/_{Y=2}\right)=\dfrac{P(X=2\cap Y=2)}{P(Y=2)}=\dfrac{\frac{2}{12}}{\frac{6}{12}}=\dfrac{2}{6}$$

$$P\left(X=3/_{Y=2}\right)=\dfrac{P(X=3\cap Y=2)}{P(Y=2)}=\dfrac{\frac{1}{12}}{\frac{6}{12}}=\dfrac{1}{6}$$

$\left(X/_{Y=2}\right)$	0	1	2	3	
$P\left(X/_{Y=2}\right)$	0	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\Rightarrow E\left(X/_{Y=2}\right)=\frac{3}{6}+\frac{4}{6}+\frac{3}{6}=\frac{10}{6}$

Vay ta co bang ham ly thuyet mau rôi rac $E\left(X/_{Y}\right)$:

Y	0	1	2
$E\left(X/_{Y}\right)$	0	$\frac{8}{5}$	$\frac{10}{6}$

g/ Tìm hàm hồi quy mẫu của Y theo X: y=0.35714x + 0.880952

So sanh cac gia trò cua ham ly thuyet va gia trò cua ham mau.

x	0	1	2	3
Ly thuyet	0	1.6	1.4	2
Mau	.88095	1.23809	1.59523	1.95237

h/ Lap bang phan phoi T=X+Y, tính ky vong va phöông sai cua T.

Ta chæ tính cho 6 o khac 0 ma thoi.

(X,Y)	(0.0)	(1.1)	(1.2)	(2.1)	(2.2)	(3.2)
P(X,Y)	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

T	0	2	3	4	5
P(T)	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{6}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

$$E(T=X+Y)=E(X)+E(Y)=\frac{18}{12}+\frac{17}{12}=\frac{35}{12}$$

$$\mathbf{var}(T=X+Y)=\mathbf{var}(X)+\mathbf{var}(Y)+2\mathbf{cov}(X,Y)=\frac{84}{144}+\frac{59}{144}+2\frac{30}{144}=\frac{203}{144}$$

Ta coù theå thöû laïi 2 keát quaû naøy baèng tröïc tieáp theá soá vaøo baûng treân.

i/ $Z=qX+(1-q)Y$, ñeå söï ñaàu tö naøy laø ít ruûi ro nhaát. töùc ñoä phaân taùn (phöông sai) cuûa noù laø cöïc tieåu.

$$\mathbf{var}(Z)=q^2\,\mathbf{var}(X)+(1-q)^2\,\mathbf{var}(Y)+2q(1-q)\mathbf{cov}(X,Y)=$$

$= q^2\frac{84}{144}+(1-q)^2\frac{59}{144}+2q(1-q)\frac{30}{144}$, laáy ñaïo haøm, cho baèng 0, ñaïo haøm baäc 2 döông, neân coù cöïc tieåu:

$$\left[\mathbf{var}(Z)\right]_q^{/}=\left(83q^2-58q+59\right)_q^{/}=2\times 83q-58=0\Rightarrow q=\frac{29}{83}=35\%$$

nghóa laø neân ñaàu tö cho chæ tieâu X laø 35% vaø chæ tieâu Y laø 65%.

Toång quaùt: Neáu:

- $Z=qX+(1-q)Y$ thì q toái öu laø $q=\dfrac{s_Y^2-\mathbf{cov}(XY)}{s_X^2+s_Y^2-2\,\mathbf{cov}(XY)}$
- $Z=qY+(1-q)X$ thì q toái öu laø $q=\dfrac{s_X^2-\mathbf{cov}(XY)}{s_X^2+s_Y^2-2\,\mathbf{cov}(XY)}$ ¢

CHÖÔNG 3: Öôùc löôïng khoaûng

2.1. Ñònh nghóa

Ngöôøi ta goïi $\left[\hat{q}_1;\hat{q}_2\right]$ hay $[q_1;q_2]$ laø khoaûng tin caäy vôùi ñoä tin caäy $(1-a)$ cho tröôùc neáu:

$$P\left[\left|q-\hat{q}\right|\le e\right]=P\left[\hat{q}-e=\hat{q}_1\le q\le\hat{q}_2=\hat{q}+e\right]=1-a$$

Trong ñoù $\hat{q}_1,\hat{q}_2$ laø caùc öôùc löôïng ñieåm cuûa q .

$\hat{q}_2-\hat{q}_1=2e$, e laø ñoä chính xaùc cuûa öôùc löôïng.

a nhoû baèng 1% ñeán 5%, ñöôïc goïi laø möùc yù nghóa.

Baøi toaùn tìm khoaûng tin caäy cuûa q laø baøi toaùn öôùc löôïng khoaûng.

Tröôùc khi ñi vaøo tìm caùc khoaûng öôùc löôïng cho caùc ñaëc tröng thoáng keâ, ta caàn nhaéc laïi kieán thöùc ôû chöông 2: Cho $X_n\in N\left(m,\ s^2\right)$, vaø cho tröôùc ñoä tin caäy $(1-a)$, ta tìm ñöôïc t_a (tra baûng B) vaø $\boxed{e=s.t_a}$ goïi laø ñoä chính xaùc, sao cho:

$$P\left[\left|X_n-m\right|\le t_a.s\right]=1-a=2j\left(t_a\right)$$

$$P\left[X_n-t_a s\le m\le X_n+t_a s\right]=1-a. \qquad (*)$$

Vaäy khoaûng tin caäy laø $\left[X_n-t_a s\,,\,X_n+t_a s\right]$ (**)

trong ñoù: $m_1 = X_n - t_a s$, $m_2 = X_n + t_a s$

Baây giôø ta xeùt cuï theå caùc baøi toaùn tìm öôùc löôïng khoaûng sau:

2.2. Khoaûng tin caäy cho tæ leä ñaùm ñoâng p

Baøi toaùn

Vôùi tæ leä p caùc phaàn töû coù tính chaát A cuûa ñaùm ñoâng chöa bieát vaø ñoä tin caäy $1-a$ cho tröôùc, khoaûng tin caäy cho p kí hieäu ñôn giaûn laø $[p_1; p_2]$ thoûa (ñuùng ra laø kí hieäu $[\P_1; \P_2]$):

$$P[p_1 \le p \le p_2] = 1-a \ .$$

Vì $F_n \in N\left(p, \frac{pq}{n}\right)$, trong ñoù: $s = \sqrt{\frac{pq}{n}} \approx \sqrt{\frac{F_n(1-F_n)}{n}}$

vaø khi ñoù ô (*): $P\left(|F_n - p| \le e = t_a \sqrt{\frac{F_n(1-F_n)}{n}}\right) = 1-a$

Trong thöïc haønh ta laáy: $f_n = F_n = \frac{m}{n}$,

$p \approx f_n, \quad q \approx (1-f_n)$ vaø $s^2 = \frac{pq}{n} \Rightarrow s = \sqrt{\frac{pq}{n}} = \sqrt{\frac{f_n(1-f_n)}{n}}$

theá taát caû vaøo (**) ta coù khoaûng tin caäy laø:

$$\left[f_n - t_a\sqrt{\frac{f_n(1-f_n)}{n}}, \quad f_n + t_a\sqrt{\frac{f_n(1-f_n)}{n}}\right] = [\P_1; \P_2]$$

Trong ñoù t_a tìm ñöôïc töø $1-a = 2j(t_a)$ (tra baûng B).

Chuù yù

+ Ñoä chính xaùc cuûa öôùc löôïng laø: $e = t_a \sqrt{\frac{f_n(1-f_n)}{n}}$. Töø ñaây ta tìm ñöôïc kích thöôùc maãu:

+ $n = \left[\frac{t_a^2}{e^2} f_n(1-f_n)\right] + 1$ vaø $p \approx f_n, \quad q \approx (1-f_n)$ (vôùi n lôùn).

Ví duï1:

Trong ñôït baàu cöû toång thoáng, ngöôøi ta phoûng vaán ngaãu nhieân 1600 cöû tri thì ñöôïc bieát coù 960 ngöôøi seõ baàu cöû öùng vieân A. Vôùi ñoä tin caäy 99%, haõy xem öùng vieân A truùng cöû khoâng? Bieát tæ leä quaù baùn laø thaéng cöû. Sau ñoù laøm laïi vôùi ñoä tin caäy 95%.

Giaûi:

n=1600, Öôùc löôïng khoaûng cho tæ leä p cuûa phieáu baàu cho öùng vieân A: $f_n = \frac{960}{1600} = 0.6$,

$1-a = 0.99 = 2j(t_a) \Rightarrow j(t_a) = 0.495 \Rightarrow t_a = 2.58$,

ñaây laø öôùc löôïng tæ leä, neân ta coù

$F_n \in N\left(p, s^2 = \frac{pq}{n}\right) \Rightarrow s = \sqrt{\frac{pq}{n}} = \sqrt{\frac{f_n(1-f_n)}{n}}$ ==>

$e = s \times t_a == \sqrt{\frac{f_n(1-f_n)}{n}} \times t_a = \sqrt{\frac{0.6 \times 0.4}{1600}} \times 2.58$ =0.0315

$\left|p - f_n\right| \le e \Leftrightarrow f_n - e \le p \le f_n + e$, ta quan tam ñen ve trai toi thieu la:

$f_n - e \le p \Leftrightarrow 0.6 - 0.0315 = 0.568 > 0.5$.

(tæ le qua ban la thang cö 0.5)

Vôi ño tin cay 99%, tæ le phieu bau toi thieu cho öng vien A la 56.8%.   Vay vôi ño tin cay 99%. Öng vien A trung cö.

Sinh vien tö lam lai vôi ño tin cay 95% ¢

Ví du2:

Ñe öôc löông so hai cau tren 1 hon ñao ngööi ta ñeo vong cho 2000 con. Sau mot thôi gian bat lai 400 con thay co 80 con ñeo vong. Hay öôc löông so hai cau vôi ño tin cay 95%, 99%.

Giai:

Neu dö ñoan tho:	400 con-----	80	ñeo vong
	?	2000	ñeo vong

Ta co: $2000 \times \dfrac{400}{80} = 2000 \times \dfrac{1}{p} = 10000$ con, trong ño

$$p = f_n = \frac{80}{400} = 0.2$$

Tính vôi ño tin cay 95% nhö sau: Ta öôc löông tæ le p ñam ñong

$$f_n = \frac{80}{400} = 0.2,\ P\left[\left|p - f_n\right| \le e = \frac{\sqrt{f_n(1 - f_n)}}{\sqrt{n}}.t_a\right] = 1 - a = 2j\left(t_a\right)$$

$$1 - a = 0.95 = 2j\left(t_a\right) \Rightarrow j\left(t_a\right) = 0.475 \Rightarrow t_a = 1.96$$

$$e = \frac{\sqrt{f_n(1 - f_n)}}{\sqrt{n}} \times t_a = \frac{\sqrt{0.2 \times 0.8}}{\sqrt{400}} \times 1.96 = 0.0392$$

$$\left|p - f_n\right| \le e \Leftrightarrow f_n - e \le p \le f_n + e \Leftrightarrow 0.2 - 0.0392 \le p \le 0.2 + 0.0392$$

<=> $p_1 = 0.1608 \le p \le 0.2392 = p_2,\quad p_1 = \dfrac{n}{N_1} = \dfrac{2000}{N_1},\quad p_2 = \dfrac{n}{N_2} = \dfrac{2000}{N_2},$

Vay $N_2 = \dfrac{2000}{p_2} \le N \le \dfrac{2000}{p_1} = N_1$

$\Leftrightarrow 8362 = \dfrac{2000}{0.1608} \le N \le \dfrac{2000}{0.2392} = 12438$ .Vay $8362 \le N \le 12438$

Tính vôi ño tin cay 99% $\Rightarrow t_a = 2.58$,

$e = \dfrac{\sqrt{0.2 \times 0.8}}{\sqrt{400}} \times 2.58 = 0.0516 \Rightarrow$ 0.1484 $= 0.2 - 0.0516 \le p \le 0.2 + 0.0516 =$0.2516

Vay $7949 = \dfrac{2000}{0.2516} \le N \le \dfrac{2000}{0.1484} = 13477.$

nhan xet: khi ño tin cay (xac suat) cao thì khoang rôi rong hôn, nghia la ño chính xac e tang. $[8362, 12438] \subset [7949,\ 13477]$ ¢

2.3. Öôc löông trung bình ñam ñong m

Bai toan

Gia sö ñam ñong co trung bình m chöa biet. Can cö vao ($X_1,\dots,X_n$), tìm mot khoang $\left[m_1(X_1,\dots,X_n);\ m_2(X_1,\dots,X_n)\right]$ sao cho

$$P\left[m_1(X_1,\dots,X_n)\le m\le m_2(X_1,\dots,X_n)\right]=1-a.$$

Tröông hôp 1:

\+ $n\ge 30,\ s^2$ ña biet $\Rightarrow \overline{X}_n\in N\left(m,\ \dfrac{s^2}{n}\right)$.

Cho tröôc $1-a$, ta tìm ñöôc t_a (bang B) sao cho

$$P\left[\left|\overline{X}_n-m\right|\le\frac{s}{\sqrt{n}}.t_a\right]=1-a=2j\left(t_a\right)$$

$$P\left[\overline{X}_n-t_a\frac{s}{\sqrt{n}}\le m\le\overline{X}_n+t_a\frac{s}{\sqrt{n}}\right]=1-a.$$

Vay $m_1=\overline{X}_n-t_a\dfrac{s}{\sqrt{n}},\qquad m_2=\overline{X}_n+t_a\dfrac{s}{\sqrt{n}}.$

Tröông hôp 2:

\+ $n\ge 30,\ s^2$ chöa biet. Thay s^2 bôi S^2 ta co

$$m_1=\overline{X}_n-t_a\frac{S}{\sqrt{n}},\qquad m_2=\overline{X}_n+t_a\frac{S}{\sqrt{n}}.$$

Tröông hôp 3:

\+ $n<30,\ s^2$ ña biet, X co phan phoi chuan thì

$$m_1=\overline{X}_n-t_a\frac{s}{\sqrt{n}},\qquad m_2=\overline{X}_n+t_a\frac{s}{\sqrt{n}}.$$

Tröông hôp 4:

\+ $n<30,\ s^2$ chöa biet, X co phan phoi chuan.

Khi ño $\dfrac{\overline{X}_n-m}{\frac{S}{\sqrt{n}}}$ co phan phoi Student (n – 1) bac tö do, biet $1-a$, ta tìm ñöôc t_a^{n-1} (bang C phan phoi Student) sao cho:

$$P\left[\left|t_{n-1}\right|=\left|\frac{\overline{X}_n-m}{\frac{S}{\sqrt{n}}}\right|\le t_a^{n-1}\right]=P\left[\left|\overline{X}_n-m\right|\le t_a^{n-1}\frac{S}{\sqrt{n}}=e\right]=1-a\ ,$$

vôi $e=t_a^{n-1}s=t_a^{n-1}\dfrac{S}{\sqrt{n}}$

Suy ra: $m_1=\overline{X}_n-t_a^{n-1}\dfrac{S}{\sqrt{n}},\qquad m_2=\overline{X}_n+t_a^{n-1}\dfrac{S}{\sqrt{n}}.$

Phan tích: $t=\dfrac{\overline{X}_n-m}{\frac{S}{\sqrt{n}}}=\dfrac{\frac{\overline{X}_n-m}{\frac{s}{\sqrt{n}}}}{\sqrt{\frac{V}{n-1}}}=\dfrac{Z}{\sqrt{\frac{V}{n-1}}},$

trong ño, ña biet :

$$Z = \frac{\overline{X}_n - m}{\frac{s}{\sqrt{n}}} \in N(0,1), \quad V = \frac{(n-1)S^2}{s^2} \in c^2(n-1)$$

vay theo ñònh nghóa 2 cua phan phoi Student ta co $t \in t(n-1)$.

<u>Tom lai:</u>

1/ $n \geq 30,\ s^2$ ña biet

+ Tính $\overline{x_n}$.

+ Tö $1-a \Rightarrow \frac{1-a}{2} \xrightarrow{B} t_a$.

+ Khoang tin cay $\left(\overline{x_n} - e,\ \overline{x_n} + e\right)$ vôi ño chính xac: $e = t_a \frac{s}{\sqrt{n}}$.

2/ $n \geq 30,\ s^2$ chöa biet

+ Tính $\overline{x_n},\ \hat{s}^2 \Rightarrow s^2 = \frac{n}{n-1}\hat{s}^2$.

+ Tö $1-a \Rightarrow \frac{1-a}{2} \xrightarrow{B} t_a$ (tra bang B)

+ Khoang tin cay $\left(\overline{x_n} - e,\ \overline{x_n} + e\right)$ vôi ño chính xac: $e = t_a \frac{s}{\sqrt{n}}$.

3/ $n < 30, X \in N\left(m, s^2\right), s^2$ ña biet (nhö tröông hôp 1).

4/ $n < 30, X \in N\left(m, s^2\right), s^2$ chöa biet

+ Tính $\overline{x_n},\ \hat{s}^2 \Rightarrow s^2 = \frac{n}{n-1}\hat{s}^2$.

+ Tö $1-a \Rightarrow a \xrightarrow{C} t_a^{n-1}$ (tra bang C)

+Khoang tin cay $\left(\overline{x_n} - e,\ \overline{x_n} + e\right)$ vôi ño chính xac: $e = t_a^{n-1} \frac{s}{\sqrt{n}}$.

<u>Chu y</u>

Khi co mau cu the, ta thay $\overline{X_n}$ bôi $\overline{x_n}$, S^2 bôi s^2.

--Một số ñề thi mẫu

Trường ĐHCN Tp.HCM

Bài 1. (3đ) Trong R^3, với cơ sở chính tắc, cho $\alpha_1=(1, -2, -3)$, $\alpha_2=(1, -1, 1)$, $\alpha_3=(-2, -1, 3)$.
a/ Cho biết $F=\{a_1, a_2, a_3\}$ và x=(3, -3, -14), tìm $[x]_F$
b/Cho biết $B=\{a_1, a_2\}$, là cơ sở của $W=\langle a_1, a_2\rangle$, và y=(1, 2, 13) $\in W$, tìm $[y]_B$
c/Cho F= < $\alpha_1=(1, -2, -3)$, $\alpha_2=(1, -1, 1)$, y=(1, 2, 13) >, G= < (3, -2, -3), (1, 2, 4) >
Tìm một cơ sở của $F \cap G$

Bài 2. (1đ) Trong R^4, với cơ sở $B=\{\vec{u_1}, \vec{u_2}\}$ của $W=\langle \vec{u_1}, \vec{u_2}\rangle = \langle \vec{u_1}=(3, 1, 1, -2), \vec{u_2}=(3, -1, 2, -3)\rangle$. Cho biết một cơ sở khác của W là $B'=\{\vec{v_1}, \vec{v_2}\}=\{\vec{v_1}=(3, 5, -1, 0), \vec{v_2}=(6, -4, 5, -7)\}$. Cho x $\in$ W thỏa: $[x]_B=[-3 \quad 5]^T$. Tìm $[x]_{B'}$.

Bài 2. Trong R^4, với cơ sở $B=\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ của $W=\langle \vec{u_1}, \vec{u_2}, \vec{u_3}\rangle = \langle \vec{u_1}=(1,1,-3,-2), \vec{u_2}=(2,-2,0,-3), \vec{u_3}=(-1,1,1,-2)\rangle$. Cho biết một cơ sở khác của W là $B'=\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}=\{\vec{v_1}=(-5, 11, -7, -1), \vec{v_2}=(-1, 7, -11, 7), \vec{v_3}=(2, -8, 8, 2)\}$.

Cho x $\in$ W thỏa: $[x]_B=[-1 \quad 2 \quad 3]^T$. Tìm $[x]_{B'}$.

Bài 3. (1đ) Trong cơ sở chính tắc. Cho dạng tòan phương (mặt quadric) trong R^3:
$f(x,y,z)=1+16x-6y-10y^2+32xz-5z^2+10xy+46z+x^2-26yz$

Bài 5. (2đ) Tìm các điểm cực trị của hàm: (cuc tieu)
$f(x,y,z)=6x^2+22y^2+9z^2-2xy-12xz-14yz+34y-12z+10$
và các giá trị hàm tương ứng với các điểm cực trị trên.

Bài 5. (2đ) Tìm các điểm cực trị của hàm: (cuc Dai)
$f(x,y,z)=1931+4x+20y-22yz+28xy+2xz-10x^2+38z-26y^2-15z^2$
và các giá trị hàm tương ứng với các điểm cực trị trên.

Bài 6. (1đ). Cho hàm $f(x_1, x_2)=5x_1^2-6x_1x_2+20x_1+18x_2^2-102x_2$, với ràng buộc

$\varphi(x_1, x_2)=5x_1-12x_2+75=0$. Tìm các điểm cực trị của $f(x_1, x_2)$ và các giá trị hàm tương ứng với các điểm cực trị trên.

Bài 6. (1đ). Cho hàm $f(x_1, x_2)=3x_1-4x_2$, với ràng buộc

$\varphi(x_1, x_2)=8x_1x_2+16x_1-8x_2^2+24x_2+104=0$. Tìm các điểm cực trị của $f(x_1, x_2)$ và các giá trị hàm tương ứng với các điểm cực trị trên.

Bài 6. (1đ). Cho hàm $f\left(x_1,x_2\right) = -31x_1 + 17x_2 + 27 + 9x_1^2 - 12x_1x_2 + 4x_2^2$, với ràng buộc $j\left(x_1,x_2\right) = 9x_1^2 - 12x_1x_2 - 11x_1 + 4x_2^2 + 11x_2 + 2 = 0$. Tìm các điểm cực trị của $f\left(x_1,x_2\right)$ và các giá trị hàm tương ứng với các điểm cực trị trên.

Bài 2. (2 đ) Cho hàm mật độ đồng thời của X và Y:

$$f(x,y) = \begin{cases} ky^2x, & 0 < x < 2y \le 4 \\ 0, & \text{noi khac} \end{cases}$$

a) Tìm k. Tính kỳ vọng E(Y).

b) Tính xác suất $P(X+Y<2)$.

c) Tìm hàm phân phối F(x,y) (nhắc lại: $F''_{xy} = f(x,y)$), rồi tính P(X<1 và Y<3/2).

(tính tích phaân, keát quaû coù theå baám maùy ra soá thaäp phaân).

Baøi 3. (2ñ). Moät ngöôøi muoán môøi 500 ngöôøi döï tieäc cöôùi, bieát raèng xaùc suaát ngöôøi ñöôïc môøi ñi döï tieäc thöïc söï laø 80%. Vaäy phaûi ñaët ít nhaát bao nhieâu choã ngoài, ñeå xaùc suaát taát caû khaùch ñöôïc môøi coù choã ngoài laø lôùn hôn 97%.

Bài 4. (4 đ). X (đơn vị: kg) và Y (đơn vị: cm) là hai chỉ tiêu của một loại sản phẩm. Điều tra một mẫu ta có bảng số liệu sau:

X \ Y	60-64	64-68	68-72	72-76
1	**5**	**8**		
2	**7**	**12**		
5		**16**	**9**	
8		**15**	**10**	5
9			8	4

a) Tính r_{YX}, cov(X,Y), var(3X-4Y).

b/ Những sản phẩm có chỉ tiêu trên 68cm là sản phẩm loại A. Ước lượng tỷ lệ sản phẩm loại A với độ tin cậy 95%.

c/ Có tài liệu nói rằng: trung bình của chỉ tiêu X của sản phẩm loại A là 5 kg. Cho nhận xét về tài liệu này với mức ý nghĩa 5%. Giả thiết X có phân phối chuẩn.

Người giới thiệu